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On Inverse Form Finding for Anisotropic Elastoplastic Materials

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Abstract. The goal of this contribution is to present an extension of the work of Germain et al. [1] on inverse form finding for anisotropic hyperelastic materials to the case of anisotropic elastoplastic materials formulated in the logarithmic strain space. A short review of the theoretical aspects is presented. Based on some numerical examples we show that our method for inverse form finding in elastoplasticity can be used if the plastic strains are previously given as it is the case when a desired hardening state is prescribed.

Keywords: Anisotropy; Logarithmic Strain; Inverse Form Finding; Large Strain; Elastoplasticity
PACS: 46.35.+z

INTRODUCTION

A challenge in the design of work pieces is the determination of the initial, undeformed shape such that under a given load a part will obtain the desired deformed shape. This problem is inverse to the standard (direct) static analysis, in which the undeformed shape is known and the deformed unknown. In [1] the method originally proposed in [2] is extended to anisotropic hyperelasticity that is based on logarithmic (Hencky) strains. In this contribution, we present an extension of our work [1] to anisotropic elastoplasticity. The logarithmic strains are decomposed into an elastic and a plastic part. The used constitutive model is based on the exploitation of the dissipation inequality. The governing equation for the resulting finite element analysis is the weak form of the balance of momentum formulated in terms of the deformed configuration using the Cauchy stress tensor. The anisotropic free energy density is decomposed into a quadratic function of the logarithmic strain and the fourth-order elasticity tensor, on which we apply a spectral decomposition for the use of different symmetry classes of anisotropic materials (Kelvin modes), and a plastic contribution. Here a von-Mises yield criterion is considered for simplicity. A radial return mapping algorithm is used to solve the plastic problem. After reviewing the pertinent theoretical aspects and based on several examples we will show that our inverse form finding model in elastoplasticity can be used if the plastic strains are previously given as it is the case when a desired hardening state is prescribed.

KINEMATICS OF GEOMETRICALLY NONLINEAR CONTINUUM MECHANICS

FIGURE 1. Material (left) and spatial configuration (right).

Let $\mathcal{B}_0$ denote the material configuration (the undeformed shape) of a continuum body parameterized by material coordinates $\mathbf{X}$ and $\mathcal{B}_t$ the corresponding spatial configuration (the deformed shape) parameterized by spatial coordinates $\mathbf{x}$, as depicted in Figure 1. In the direct problem, the material configuration is given and we seek to determine the (direct) deformation map $\boldsymbol{\varphi}$. On the contrary, in the inverse problem, the spatial configuration is given and we seek to determine the inverse deformation map $\boldsymbol{\Phi}$,

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}) : \mathcal{B}_0 \longrightarrow \mathcal{B}_t \quad \text{and} \quad \mathbf{X} = \boldsymbol{\Phi}(\mathbf{x}) : \mathcal{B}_t \longrightarrow \mathcal{B}_0. \quad (1)$$

The corresponding direct and inverse deformation gradients together with their Jacobian determinants are defined as

$$\mathbf{F} = \nabla_{\mathbf{X}} \boldsymbol{\varphi}, \quad J = \det \mathbf{F} \quad \text{and} \quad \mathbf{f} = \nabla_{\mathbf{x}} \boldsymbol{\Phi}, \quad j = \det \mathbf{f}. \quad (2)$$

Here $\nabla_{\mathbf{X}}$ denotes the gradient operator with respect to the material coordinates $\mathbf{X}$ and $\nabla_{\mathbf{x}}$ the gradient operator with respect to the spatial coordinates $\mathbf{x}$. It follows immediately from the above definitions that

$$\boldsymbol{\Phi} = \boldsymbol{\varphi}^{-1}, \quad \mathbf{f} = \mathbf{F}^{-1} \quad \text{and} \quad j = J^{-1}. \quad (3)$$

ANISOTROPIC ELASTOPLASTICITY IN LOGARITHMIC STRAIN SPACE

We assume a decomposition of the free energy function into an elastic and a plastic part

$$\begin{aligned} \Psi(\mathbf{E}^e, \alpha) &= \Psi^e(\mathbf{E}^e) + \Psi^p(\alpha) \\ &= \frac{1}{2} \mathbf{E}^e : \mathbb{E}^e : \mathbf{E}^e + \frac{1}{2} h \alpha^2 + (\sigma_{\infty} - \sigma_0) \left(\alpha + \frac{\exp(-w\alpha)}{w} \right). \end{aligned} \quad (4)$$

where $\mathbf{E}^e$ are the elastic strains, $\mathbb{E}^e$ is the fourth-order elasticity tensor, h a material parameter associated with the isotropic hardening, w a saturation parameter (defining the nonlinearity of the hardening), σ_0 the initial yield stress, σ_{∞} the infinite yield stress and α a scalar internal variable that models isotropic hardening. For cubic materials, $\mathbb{E}^e$ can be decomposed into Kelvin modes [4]:

$$\mathbb{E}^e = 3\kappa \mathbb{P}_1 + 2\mu \mathbb{P}_2 + 2E_{55} \mathbb{P}_3, \quad (5)$$

where κ is the bulk modulus, μ the shear modulus and E_{55} a constant material parameter. $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ are fourth-order projection tensors. They can be expressed in terms of the volumetric and symmetric deviatoric part of the fourth-order identity tensor

$$\mathbb{P}_1 = \mathbb{I}_{\text{vol}}, \quad (6)$$

$$\mathbb{P}_2 + \mathbb{P}_3 = \mathbb{I}_{\text{dev}}^{\text{sym}}. \quad (7)$$

We assume an additive decomposition of the plastic strains into an elastic and a plastic part in terms of the second-order logarithmic strain tensor

$$\mathbf{E} = \frac{1}{2} \ln \mathbf{C} = \mathbf{E}^e + \mathbf{E}^p. \quad (8)$$

The spectral decomposition of the right Cauchy–Green strain tensor $\mathbf{C}$ reads

$$\mathbf{C} = \mathbf{F}^t \cdot \mathbf{F} = \sum_{i=1}^3 \lambda_i \mathbf{M}_i, \quad (9)$$

where $\{\lambda_i\}_{i=1,2,3}$ are the real-valued eigenvalues of $\mathbf{C}$ and $\{\mathbf{M}_i\}_{i=1,2,3}$ the associated eigenbases [5]. The spectral representation facilitates the computation of the logarithmic strain

$$\mathbf{E} = \frac{1}{2} \sum_{i=1}^3 \ln \lambda_i \mathbf{M}_i. \quad (10)$$

The first and second derivatives of the logarithmic strain with respect to the right Cauchy–Green strain are denoted by

$$\mathbb{P} = 2 \frac{\partial \mathbf{E}}{\partial \mathbf{C}} \quad \text{and} \quad \mathbb{L} = 2 \frac{\partial \mathbb{P}}{\partial \mathbf{C}} = 4 \frac{\partial^2 \mathbf{E}}{\partial \mathbf{C} \partial \mathbf{C}}. \quad (11)$$

For further details on the computation of these derivatives the interested reader is referred to [6]. Using (11), the Piola–Kirchhoff stress may be expressed as

$$\mathbf{S} = \mathbf{T} : \mathbb{P} \quad \text{with} \quad \mathbf{T} = \mathbb{E}^e : \mathbf{E}^e. \quad (12)$$

Considering this expression, the linearization of the Piola–Kirchhoff stress (tangent operator needed in a Newton type solution scheme) reads

$$\mathbb{C} = 2 \frac{\partial \mathbf{S}}{\partial \mathbf{C}} = \mathbb{P}^T : \mathbb{E}^* : \mathbb{P} + \mathbf{T} : \mathbb{L}, \quad (13)$$

where the transposition symbol $[\bullet]^T$ refers to an exchange of the first and last pairs of indices. The fourth-order tensor $\mathbb{E}^*$ is the fourth-order elasticity tensor $\mathbb{E}^e$ in the case of an elastic loading or the elastoplasticity tensor $\mathbb{E}^{ep}$ in the case of a plastic loading [3].

DISSIPATION AND PLASTIC FLOW RESPONSE

In the logarithmic strain space the dissipation inequality can be written as

$$\mathbf{T} : \mathbf{E}^p - \frac{\partial \Psi}{\partial \alpha} \cdot \dot{\alpha} \geq 0, \quad (14)$$

where $\circ$ denotes the time derivative and $\{\mathbf{E}^p, \alpha\}$ is the set of internal variables. We consider the following quadratic yield function

$$\Upsilon = \|\text{dev} \mathbf{T}\|_{\mathbb{P}} - \sqrt{\frac{2}{3}} B \quad \text{with} \quad B = -h\alpha - (\sigma - \sigma_0)(1 - \exp(-w\alpha)). \quad (15)$$

Here we set the metric $\mathbb{P}$ to $\mathbb{I}_{\text{dev}}^{\text{sym}}$ for simplicity. Using the principle of maximum plastic dissipation and the definition of the Lagrange function $\mathcal{L}$,

$$\mathcal{L}(\mathbf{T}, \frac{\partial \Psi}{\partial \alpha}, \gamma) = -\mathbf{T} : \mathbf{E}^p - \frac{\partial \Psi}{\partial \alpha} \cdot \dot{\alpha} + \gamma(\Upsilon - \sqrt{\frac{2}{3}} \sigma_0), \quad (16)$$

we obtain the Karush-Kuhn-Tucker equations

$$\left\{ \begin{array}{l} \mathbf{E}^p = \gamma \frac{\partial \Upsilon}{\partial \mathbf{T}}, \quad \dot{\alpha} = \gamma \frac{\partial \Upsilon}{\partial B} = \sqrt{\frac{2}{3}} \gamma, \\ \gamma \geq 0, \quad \Upsilon - \sqrt{\frac{2}{3}} \sigma_0 \leq 0 \quad \text{and} \quad \gamma \Upsilon = 0, \end{array} \right. \quad (17)$$

where $\gamma \geq 0$.

DETERMINATION OF THE UNDEFORMED SHAPE FROM EQUILIBRIUM

The equilibrium statement may be expressed by the following variant of the boundary value problem, here in terms of spatial description quantities

$$\begin{aligned} \text{div} \boldsymbol{\sigma} &= \mathbf{0} & \text{in } \mathcal{B}_t, \\ \boldsymbol{\sigma} \cdot \mathbf{n} &= \bar{\mathbf{t}}_t & \text{on } \partial \mathcal{B}_t^t, \\ \boldsymbol{\varphi} &= \bar{\boldsymbol{\varphi}} & \text{on } \partial \mathcal{B}_t^\varphi. \end{aligned} \quad (18)$$

Here $\bar{\mathbf{t}}_t$ is a prescribed (given) traction per unit area in the spatial configuration, $\bar{\boldsymbol{\varphi}}$ is a prescribed boundary deformation and div denotes the divergence operator with respect to the spatial coordinates $\mathbf{x}$. The (symmetric) Cauchy stress $\boldsymbol{\sigma}$ is obtained from the (symmetric) Piola–Kirchhoff stress by a push-forward according to

$$J \boldsymbol{\sigma} = \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^t. \quad (19)$$

All quantities are parameterized in spatial coordinates $\mathbf{x}$. The weak form of the given boundary value problem reads

$$g(\Phi, \boldsymbol{\eta}; \mathbf{x}) = \int_{\mathcal{B}_t} \nabla_{\mathbf{x}} \boldsymbol{\eta} : \boldsymbol{\sigma} \, dv - \int_{\partial \mathcal{B}_t^f} \boldsymbol{\eta} \cdot \bar{\mathbf{t}}_t \, da = 0 \quad \forall \boldsymbol{\eta} \in V_t, \quad (20)$$

where $\boldsymbol{\eta} \in V_t = \{\boldsymbol{\eta} \in H^1(\mathcal{B}_t) | \boldsymbol{\eta} = \mathbf{0} \text{ on } \partial \mathcal{B}_t^p\}$ denotes the test function. Clearly, equation (20) is a virtual work statement and all integrals extend over the spatial configuration, which is assumed to be given. Since we consider the spatial coordinates $\mathbf{x}$ as fixed and since we want to determine the inverse deformation map $\mathbf{X} = \Phi(\mathbf{x})$, we need the linearization (directional derivative) of the weak form in the direction $\Delta \Phi$ at fixed spatial coordinates $\mathbf{x}$

$$\frac{d}{d\varepsilon} g(\Phi + \varepsilon \Delta \Phi, \boldsymbol{\eta}; \mathbf{x})|_{\varepsilon=0} = \int_{\mathcal{B}_t} \nabla_{\mathbf{x}} \boldsymbol{\eta} : \mathbb{a} : \Delta \mathbf{f} \, dv. \quad (21)$$

The computation of the fourth-order tangent operator $\mathbb{a}$ simplifies considerably if we make the following assumptions:

1. the surface tractions per unit area in $\partial \mathcal{B}_t^f$ are given, i.e. they are independent of the inverse deformation map,
2. the material is homogeneous, i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{f}) \neq \boldsymbol{\sigma}(\mathbf{f}, \Phi)$.

With these assumptions $\mathbb{a}$ follows in a straightforward manner from the relation between the Cauchy and the Piola–Kirchhoff stresses and application of the chain and product rules of differentiation

$$\mathbb{a} := \frac{\partial [j\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^t]}{\partial \mathbf{f}} = \boldsymbol{\sigma} \otimes \mathbf{F}^t - \mathbf{F} \bar{\otimes} \boldsymbol{\sigma} + j\mathbf{F} \cdot \left[\frac{1}{2} \mathbb{C} : \frac{\partial \mathbf{C}}{\partial \mathbf{f}} \right] \cdot \mathbf{F}^t - \boldsymbol{\sigma} \underline{\otimes} \mathbf{F}, \quad (22)$$

where the operators $\bar{\otimes}$ and $\underline{\otimes}$ are defined by $[\mathbf{A} \underline{\otimes} \mathbf{B}]_{IJKL} = A_{IL} B_{JK}$ and $[\mathbf{A} \bar{\otimes} \mathbf{B}]_{IJKL} = A_{IK} B_{JL}$.

DISCRETIZATION OF THE INVERSE PROBLEM

For the finite element analysis of the inverse problem (20) the spatial configuration $\mathcal{B}_t$ is discretized into n_{el} elements

$$\mathcal{B}_t \approx \mathcal{B}_t^h = \bigcup_{e=1}^{n_{el}} \mathcal{B}_t^e. \quad (23)$$

Following the standard isoparametric approach, the deformation maps are approximated on each element by the following shape functions

$$\mathbf{X}^e(\boldsymbol{\xi}) = \Phi^e(\boldsymbol{\xi}) = \sum_{i=1}^{n_{en}} \Phi^{(i)} N^{(i)}(\boldsymbol{\xi}) \quad \text{and} \quad \mathbf{x}^e(\boldsymbol{\xi}) = \boldsymbol{\varphi}^e(\boldsymbol{\xi}) = \sum_{i=1}^{n_{en}} \boldsymbol{\varphi}^{(i)} N^{(i)}(\boldsymbol{\xi}). \quad (24)$$

Thereby the shape functions $N^{(i)}$ are parameterized by isoparametric coordinates $\boldsymbol{\xi}$ defined on the isoparametric cube $\mathcal{B}^\xi = [-1, 1]^{n_{dim}}$, where n_{en} is the total number of nodes per element. $\Phi^{(i)}$ and $\boldsymbol{\varphi}^{(i)}$ denote nodal values. Following the Bubnov–Galerkin method the test function is approximated by the same shape functions $N^{(i)}$

$$\boldsymbol{\eta}^e(\boldsymbol{\xi}) = \sum_{i=1}^{n_{en}} \boldsymbol{\eta}^{(i)} N^{(i)}(\boldsymbol{\xi}). \quad (25)$$

Substituting the finite element approximations into the weak form, we obtain the discrete equilibrium condition as a residual that is expressed at each node (i) (n_{np} is the total number of node points) as

$$\mathbf{r}^{(i)} = \mathbf{r}_{ext}^{(i)} - \mathbf{r}_{int}^{(i)}, \quad i = 1 \dots n_{np}. \quad (26)$$

Here the contributions to the internal and external nodal forces read

$$\mathbf{r}_{int}^{(i)} = \mathbf{A} \int_{\mathcal{B}_t^e} \boldsymbol{\sigma} \cdot \nabla_{\mathbf{x}} N^{(i)} \, dv, \quad (27)$$

$$\mathbf{r}_{ext}^{(i)} = \mathbf{A} \int_{\partial \mathcal{B}_t^e} \mathbf{t}_t^e N^{(i)} \, da. \quad (28)$$

The objective of the inverse problem is to determine the undeformed shape for a given spatial configuration, thus the above residual is considered as a nonlinear function of the nodal inverse deformation map

$$\mathbf{r}^{(i)} = \mathbf{r}^{(i)}(\Phi^{(j)}), \quad i, j = 1 \dots n_{np}. \quad (29)$$

Thus the tangent stiffness matrix of the inverse problem follows as the Jacobian matrix of the residual with respect to the nodal inverse deformation map

$$\mathbf{K}^{(ij)} := -\frac{\partial \mathbf{r}^{(i)}}{\partial \Phi^{(j)}} = \mathbf{A} \int_{\mathcal{B}_t^e} \nabla_{\mathbf{x}} N^{(i)} \cdot \mathbf{a} \cdot \nabla_{\mathbf{x}} N^{(j)} dv. \quad (30)$$

In the above equation $\cdot$ denotes contraction with the second index of the corresponding tangent operator. The implementation renders for the inverse problem quadratic convergence within a Newton solution scheme.

EXAMPLES

In the following we consider a three-dimensional thick sheet (length=100, width=20 and thickness=4 (Figure 2)), which is clamped on one end and subjected to a vertical distributed force at the opposite end. We consider a cubic material with the following material and nonlinear isotropic hardening parameters: $\mu=807$, $\lambda=1210$, $E_{55}=850$, $\sigma_0=20$, $\sigma_\infty=27$, $h=2$ and $w=5$. The applied force is equal to 3 (300 loading and unloading steps).

To show that the undeformed shape is unique if the plastic strains are prescribed, we computed first the direct problem. The plastic strains and the coordinates of the deformed shape (Figure 3 (c)) obtained by the direct computation are used as start values for the computation of the inverse problem. The difference between the coordinates of the undeformed sheet used for the computation of the direct problem (Figure 2) and the coordinates of the undeformed sheet obtained by solving the inverse problem (Figure 3 (a)) is of the order of magnitude 10^{-8} .

In a second computation we took the same deformed sheet (Figure 3 (c)), but set the initial plastic strains to zero, for example. As a comparison between Figure 3 (a) and Figure 3 (b) shows, the undeformed shape of the second computation differs significantly from that of the first computation. As expected we could observe that the difference between the two inverse computations (the first with plastic strains obtained by the direct problem and the second with zero plastic strains) increases with increasing plastic strains (direct problem).

FIGURE 2. Discretization of the undeformed thick sheet in the material configuration $\mathcal{B}_0$ and nodal forces.

CONCLUSION

This work extends a three dimensional procedure for the determination of the undeformed shape of a workpiece when knowing its desired deformed shape, the boundary conditions and the loads, to the case of anisotropic elastoplastic materials, whereby a logarithmic strain space formulation is used. We simplified the formulation by some mild assumptions: The body forces are zero, the surface tractions are independent of the inverse deformation mapping and the material is homogeneous. Based on a numerical example, we showed that our method can be used if the plastic strains are prescribed, as it is the case if a desired hardening state is known.

FIGURE 3. (a) computed undeformed sheet with $\mathbf{E}^P$ from the direct problem, (b) computed undeformed sheet with $\mathbf{E}^P=0$, and (c) computed deformed sheet in the spatial configuration $\mathcal{B}_t$.

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