Highly-conductive energetic coherent interfaces subject to in-plane degradation

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Abstract

The presence of an interface can influence the thermo-mechanical response of a body. This influence is especially pronounced at small scales where the interface area to bulk volume ratio significantly increases. Since the thermo-mechanical properties of an interface can differ from those of the bulk, within interface continuum theory an interface is endowed with its own thermo-mechanical energetic structure. To date, the effects of interface in-plane damage on the thermo-mechanical response of a highly conductive interface have not been accounted for. Therefore in this contribution the computational aspects of thermo-mechanical solids containing highly conductive interfaces subject to in-plane degradation are studied.

To this end, the equations governing a fully non-linear transient problem are given. They are solved using the finite element method. The results are illustrated through a series of three-dimensional numerical examples for various interfacial parameters. In particular a comparison is made between the results of the intact and the degraded highly conductive interface formulation.

Keywords

Thermo-mechanically energetic interfaces, interface elasticity, highly conductive interfaces, nanomaterials, non-local damage, finite element method

1. Introduction

The thermo-mechanical properties of interfaces can be considerably different from those of the bulk. Such a discrepancy can influence the overall response of the material, especially at small scales where the interface area to bulk volume ratio is significantly large. There are various approaches that attempt to model thermo-mechanical energetic and thermal interfaces. Recall that a thermo-mechanically *energetic interface* is endowed with its own distinct thermo-mechanical behavior or more accurately with its own energetic structure in the form of internal energy, entropy, constitutive relations and dissipation. For the case of elastic behavior, these

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Figure 1. (a) Geometrically non-coherent interface, (b) geometrically coherent elastic interface, (c) thermo-hyperelastically energetic and thermally highly conductive interface subject to in-plane degradation. The interface in this work is thermally and kinematically coherent since no jump in temperature and geometry is allowed across the interface, $[\Theta] = 0$ and $[\varphi] = 0$, respectively. In (c), the in-plane degradation of the interface causes the degradation of mechanical and thermal properties of the interface through a tangential damage variable $\bar{D}_{\parallel}$.

interfaces are widely described by the *interface elasticity theory* proposed by Gurtin and Murdoch [1] and Murdoch [2]. We need to point out that the energetic interfaces dealt with by this theory are mechanically coherent, meaning that a discontinuity in the displacement field across the interface is not allowed. For further details see for instance, [3–12] and references therein. The effect of interface energetics on the size-dependent elastic response of the material has been thoroughly investigated recently for instance in [13–24] and references therein.

From a thermal point of view, interfaces are labeled either *perfect* or *imperfect*. A thermally perfect interface does not allow for a discontinuity in either temperature or normal heat flux across the interface (see, for instance, [5, 25] for further details). An imperfect thermal interface, on the other hand can either be *highly conductive* allowing for solely a discontinuity in normal heat flux and not in temperature [see 26–31, for further details]; or be *lowly conductive* admitting only a discontinuity in temperature and not in normal heat flux (see [26, 32–37] for further details); or be *general imperfect* permitting discontinuities in both temperature and normal heat flux (see [14, 38–43] for further details), across the interface. Note that a *thermal interface* is non-energetic and may allow for conduction along the interface. It also needs to be pointed out that a highly conductive interface is thermally coherent.

Geometrically non-coherent non-energetic interfaces are widely modeled using the cohesive zone approach, in which a traction-separation law describes the constitutive relation (see Figure 1(a)). This approach is essentially based on the fact that local material damage takes place associated with a displacement jump across the interface. Thus in this approach the interface traction is a function of the displacement jump. Analogous to energetic coherent interface, non-coherent, that is, cohesive interfaces can also be thermal. For instance, Willam et al. [44] employed a continuum damage model of the thermo-mechanical cohesive interfaces, with distinct thermal and mechanical damage variables. For further details on thermal cohesive interfaces see [45–50].

Although the interface elasticity theory proves to be a very powerful tool to capture the material behavior at the nanometer scale, and can also be used to deal with thermo-mechanically energetic interface problems subject to various thermal assumptions (highly conductive, lowly conductive and general imperfect), it suffers from the fact that the interface behavior remains (thermo-hyper)elastic regardless of the strain level at the interface (see Figure 1(b)).

In this contribution therefore, to tackle this issue, a highly conductive energetic interface is subject to in-plane degradation modeled by a continuum damage approach. In doing so, a tangential (in-plane) damage variable $\bar{D}_\parallel$ is used to measure the degradation level at the interface (see Figure 1(c)). The variable is a function of non-local equivalent distortion, which in turn depends on the effective (undamaged) interface thermo-mechanical Helmholtz energy. This dependence suggests that both interface deformation and temperature influence the damage evolution. The non-localization is of integral-type and is employed to avoid pathological mesh-dependency associated with local damage models. In summary, the key contributions of this work are as follows:

- to derive the governing equations of a thermo-mechanical solid possessing highly conductive energetic interface subject to in-plane degradation, within the fully-nonlinear three dimensional setting;
- to present a thermodynamically consistent formulation and derive the dissipation inequality on the interface;
- to account for the effects of in-plane damage on thermo-mechanical properties of the interface;
- to derive the thermal and mechanical weak forms;
- to derive the consistent tangent stiffness matrices in the bulk and on the interface;
- to present details of the computation of solids possessing geometrically and thermally coherent energetic highly conductive interfaces within the three-dimensional, non-linear and transient setting;
- to illustrate the theory with the help of numerical examples using the finite element method.

1.1. Organization of this manuscript

This manuscript is organized as follows. First the notation and certain key concepts are briefly introduced. Section 2 summarizes the kinematics of non-linear continuum mechanics. The local governing equations including the additional contributions from the interface, together with the constitutive relations are given in Section 3. A numerical framework for the interface is established in Section 4. The framework includes the weak formulation of the governing equations, the corresponding finite element implementation and the derivation of the consistent stiffness matrices. A series of numerical examples, based on the finite element approximation of the weak form, is presented in Section 5 to elucidate the theory. Section 6 concludes this work.

1.2. Notation and definitions

Direct notation is adopted throughout. Occasional use is made of index notation, the summation convention for repeated indices being implied. The three-dimensional Euclidean space is denoted by $\mathbb{E}^3$. The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \cdot \mathbf{b} = [\mathbf{a}]_i [\mathbf{b}]_i$. The scalar product of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} : \mathbf{B} = [\mathbf{A}]_{ij} [\mathbf{B}]_{ij}$. The composition of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \cdot \mathbf{B}$, is a second-order tensor with coefficients $[\mathbf{A} \cdot \mathbf{B}]_{ij} = [\mathbf{A}]_{im} [\mathbf{B}]_{mj}$. The two non-standard products of a fourth-order tensor $\mathbb{C}$ and a vector $\mathbf{b}$ are defined by $[\mathbf{b} \cdot \mathbb{C}]_{ikl} = [\mathbb{C} \cdot \mathbf{b}]_{ikl} = [\mathbb{C}]_{ijkl} [\mathbf{b}]_j$. The non-standard minor transpose of a fourth-order tensor $\mathbb{C}$ is defined by $[\mathbb{C}]^{ijkl} = [\mathbb{C}]_{jikl}$. The action of a second-order tensor $\mathbf{A}$ on a vector $\mathbf{a}$ is given by $[\mathbf{A} \cdot \mathbf{a}]_i = [\mathbf{A}]_{ij} [\mathbf{a}]_j$. The standard product of a fourth-order tensor $\mathbb{C}$ and a second-order tensor $\mathbf{A}$ is defined by $[\mathbb{C} : \mathbf{A}]_{ij} = [\mathbb{C}]_{ijkl} [\mathbf{A}]_{kl}$. The dyadic product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a second-order tensor $\mathbf{D} = \mathbf{a} \otimes \mathbf{b}$ with $[\mathbf{D}]_{ij} = [\mathbf{a}]_i [\mathbf{b}]_j$. Two non-standard dyadic products of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ are the fourth-order tensors $[\mathbf{A} \bar{\otimes} \mathbf{B}]_{ijkl} = [\mathbf{A}]_{ik} [\mathbf{B}]_{jl}$ and $[\mathbf{A} \underline{\otimes} \mathbf{B}]_{ijkl} = [\mathbf{A}]_{il} [\mathbf{B}]_{jk}$. The average and jump of a quantity $\{\bullet\}$ over an interface are defined by $\{\{\bullet\}\} = \frac{1}{2}[\{\bullet\}^+ + \{\bullet\}^-]$ and $\llbracket \{\bullet\} \rrbracket = \{\bullet\}^+ - \{\bullet\}^-$, respectively. All over-lined quantities correspond to the interface. Table 1 gathers a list of notations frequently used in this manuscript.

2. Problem definition

This section summarizes the kinematics of non-linear continuum mechanics including highly conductive energetic interfaces and introduces the notation adopted here. Further details on the kinematics of deformable interfaces can be found in Javili et al. [27].

Consider a continuum body $\mathcal{B}$ that takes the material configuration $\mathcal{B}_0 \subset \mathbb{E}^3$ at time $t = 0$, and the spatial configuration $\mathcal{B}_t$ at $t > 0$, as depicted in Figure 2. The body $\mathcal{B}$ is partitioned into two disjoint subdomains, $\mathcal{B}_0^+$ and $\mathcal{B}_0^-$, by an interface $\mathcal{I}_0$, and thus the bulk is defined by $\mathcal{B}_0 := \mathcal{B}_0^+ \cup \mathcal{B}_0^-$, with reference placements of material particles labeled $\mathbf{X}$. The two sides of the interface $\mathcal{I}_0$ are denoted $\mathcal{I}_0^+ := \partial \mathcal{B}_0^+ \cap \mathcal{I}_0$ and

Table 1. List of important notations. Over-lined quantities $\{\bar{\bullet}\}$ correspond to the interface.

$\mathbf{F}$	bulk material deformation gradient	$\bar{\mathbf{F}}$	interface material deformation gradient
$\boldsymbol{\varphi}$	bulk deformation map	$\bar{\boldsymbol{\varphi}}$	interface deformation map
$\mathbf{X}$	bulk material coordinates	$\bar{\mathbf{X}}$	interface material coordinates
$\mathbf{x}$	bulk spatial coordinates	$\bar{\mathbf{x}}$	interface spatial coordinates
Θ	bulk temperature	$\bar{\Theta}$	interface temperature
Θ_0	bulk initial temperature	$\bar{\Theta}_0$	interface initial temperature
$\mathbf{N}$	bulk material normal to surface	$\bar{\mathbf{N}}$	interface material normal to interface
$\mathbf{n}$	bulk spatial normal to surface	$\bar{\mathbf{n}}$	interface spatial normal to interface
Ψ	bulk Helmholtz energy	$\bar{\Psi}$	interface nominal Helmholtz energy
Ξ_s	bulk specific entropy	$\bar{\Xi}_s$	interface specific entropy
Ξ	bulk entropy	$\bar{\Xi}$	interface nominal entropy
$\mathbf{P}$	bulk Piola stress	$\bar{\mathbf{P}}$	interface nominal Piola stress

Only interface			
$\boldsymbol{\varphi}^\pm$	deformation maps of $\pm$ side	$\mathbf{x}^\pm$	spatial coordinates of $\pm$ side
$\tilde{\mathbf{n}}$	spatial normal to interface boundary	$\tilde{\mathbf{N}}$	material normal to interface boundary
$\bar{\Psi}^0$	undamaged Helmholtz energy	$\bar{D}_\parallel$	damage variable
$\bar{\mathbf{P}}_0$	undamaged Piola stress tensor	$\bar{\Xi}^0$	undamaged entropy
$\bar{F}_{\text{loc}}$	local equivalent distortion	$\bar{F}_{\text{nloc}}$	non-local equivalent distortion
$\bar{F}_0$	elastic limit	$\bar{F}_{\text{max}}$	maximum attained $\bar{F}_{\text{nloc}}$

Figure 2. The bulk domain $\mathcal{B}_0$, the bulk subdomains $\mathcal{B}_0^\pm$, the interface $\mathcal{I}_0$, the two sides of the interface $\mathcal{I}_0^\pm$ and the unit normals to the surface $\mathbf{N}$, the interface $\bar{\mathbf{N}}$ and boundary of the interface $\tilde{\mathbf{N}}$, all defined in the material configuration. The bulk, interface and the two sides of interface deformation maps, denoted as $\boldsymbol{\varphi}$, $\bar{\boldsymbol{\varphi}}$ and $\boldsymbol{\varphi}^\pm$, respectively, map the material configuration to the spatial configuration at time t . The bulk domain $\mathcal{B}_t$, the bulk subdomains $\mathcal{B}_t^\pm$, the interface $\mathcal{I}_t$ and its two sides $\mathcal{I}_t^\pm$, the unit normals to the surface $\mathbf{n}$, interface $\bar{\mathbf{n}}$, and boundary of the interface $\tilde{\mathbf{n}}$, all defined in the spatial configuration. The bulk temperatures on plus and minus side of the interface and the interface temperature are denoted by Θ^+ , Θ^- and $\bar{\Theta}$, respectively. The interface unit normal is pointing from the negative side of the interface to the positive side. The bulk and (rank-deficient) interface deformation gradients are $\mathbf{F}$ and $\bar{\mathbf{F}}$, respectively.

$\mathcal{I}^- := \partial\mathcal{B}_0^- \cap \mathcal{I}_0$. The material particles on the interface are labeled $\bar{\mathbf{X}}$. The outward unit normal to $\partial\mathcal{B}_0$ is denoted $\mathbf{N}$. The outward unit normal to the boundary of the interface $\partial\mathcal{I}_0$ and tangent to the interface $\mathcal{I}_0$ is denoted $\tilde{\mathbf{N}}$. The unit normal to $\mathcal{I}_0$ is denoted $\bar{\mathbf{N}}$ whose direction is conventionally taken to point from the negative side of the interface to the positive side. The spatial counterparts of the various unit normals are $\mathbf{n}$, $\tilde{\mathbf{n}}$ and $\bar{\mathbf{n}}$, respectively. The deformation maps of the bulk, and the negative and positive sides of the interface are denoted $\boldsymbol{\varphi}$, $\boldsymbol{\varphi}^-$ and $\boldsymbol{\varphi}^+$, respectively. The restriction of the motion $\boldsymbol{\varphi}$ to the interface is defined by $\bar{\boldsymbol{\varphi}}$. The current placements of particles in the bulk and on the two sides of the interface are denoted $\mathbf{x}$ and $\mathbf{x}^\mp$, where the spatial placement of particles on the interface are designated as $\bar{\mathbf{x}}$. One should note that $\bar{\boldsymbol{\varphi}} = \boldsymbol{\varphi}^+ = \boldsymbol{\varphi}^-$ and $\bar{\mathbf{x}} = \mathbf{x}^+ = \mathbf{x}^-$, for geometrically coherent interfaces. Since the interface is of the highly conductive type the interface temperature is the same as the temperatures across the interface, $\bar{\Theta} = \Theta^+ = \Theta^-$.

Table 2. Localized force, energy and entropy balances in the bulk and on the interface in the material configuration. The notation $\{\bullet\}^p$ is to denote prescribed quantities.

Force balance	$\text{Div} \mathbf{P} + \mathbf{B}^p = \mathbf{0}$ $\overline{\text{Div}} \mathbf{\bar{P}} + \mathbf{\bar{B}}^p = -[\![\mathbf{P}]\!] \cdot \mathbf{\bar{N}}$	in $\mathcal{B}_0$ on $\mathcal{I}_0$	$\widehat{\mathbf{B}}^p = \mathbf{P} \cdot \mathbf{N}$ $\widetilde{\mathbf{B}}^p = \mathbf{\bar{P}} \cdot \mathbf{\bar{N}}$	on $\partial \mathcal{B}_0^N$ on $\partial \mathcal{I}_0^N$
Energy	$-\mathbf{P} : \text{Grad} \mathbf{V} + \text{Div} \mathbf{Q} - \mathcal{Q}^p + \text{D}_t \mathcal{E} = 0$ $-\mathbf{\bar{P}} : \overline{\text{Grad}} \mathbf{\bar{V}} + \overline{\text{Div}} \mathbf{\bar{Q}} - \overline{\mathcal{Q}}^p + \text{D}_t \overline{\mathcal{E}} = -[\![\mathbf{Q}]\!] \cdot \mathbf{\bar{N}}$	in $\mathcal{B}_0$ on $\mathcal{I}_0$	$\widehat{\mathcal{Q}}^p = -\mathbf{Q} \cdot \mathbf{N}$	on $\partial \mathcal{B}_0^N$
Entropy	$\text{Div} \mathbf{H} - H^p + \text{D}_t \mathcal{E} \geq 0$ $\overline{\text{Div}} \mathbf{\bar{H}} - \overline{H}^p + \text{D}_t \overline{\mathcal{E}} \geq -[\![\mathbf{H}]\!] \cdot \mathbf{\bar{N}}$	in $\mathcal{B}_0$ on $\mathcal{I}_0$	$\widehat{H}^p = -\mathbf{H} \cdot \mathbf{N}$ $\widetilde{H}^p = -\mathbf{\bar{H}} \cdot \mathbf{\bar{N}}$	on $\partial \mathcal{B}_0^N$ on $\partial \mathcal{I}_0^N$
$\mathbf{B}^p$	force vector per unit volume	$\widehat{\mathbf{B}}^p$	surface traction per unit area	
$\mathbf{\bar{B}}^p$	force vector per unit area	$\widetilde{\mathbf{B}}^p$	curve traction per unit length	
$\mathbf{Q}$	bulk heat flux vector per unit area	$\widehat{\mathbf{Q}}$	interface heat flux vector per unit length	
$\mathbf{H}$	bulk entropy flux vector per unit area	$\widehat{\mathbf{H}}$	interface entropy flux vector per unit length	
$\mathcal{Q}^p$	bulk heat source per unit volume	$\widehat{\mathcal{Q}}^p$	surface heat source per unit area	
$\overline{\mathcal{Q}}^p$	interface heat source per unit area	$\widetilde{\mathcal{Q}}^p$	curve heat source per unit length	
H^p	bulk entropy source per unit volume	$\widehat{H}^p$	surface entropy source per unit area	
$\overline{H}^p$	interface entropy source per unit area	$\widetilde{H}^p$	curve entropy source per unit length	
$\mathcal{E}$	bulk internal energy per unit volume	$\overline{\mathcal{E}}$	interface internal energy per unit area	

Balance of angular momentum results in the symmetry of the bulk Cauchy stress, i.e. $\mathbf{P} \cdot \mathbf{F}^t = \mathbf{F} \cdot \mathbf{P}^t$ and the interface Cauchy stress, i.e. $\mathbf{\bar{P}} \cdot \mathbf{\bar{F}}^t = \mathbf{\bar{F}} \cdot \mathbf{\bar{P}}^t$ in the material configuration.

The bulk and the (rank-deficient) interface deformation gradients $\mathbf{F}$ and $\mathbf{\bar{F}}$, together with the corresponding velocities $\mathbf{V}$ and $\mathbf{\bar{V}}$ are, respectively, defined by

$$\begin{aligned} \mathbf{F}(\mathbf{X}, t) &:= \text{Grad} \boldsymbol{\varphi}(\mathbf{X}, t), \quad \mathbf{V} := \text{D}_t \boldsymbol{\varphi}(\mathbf{X}, t) \text{ and} \\ \mathbf{\bar{F}}(\mathbf{\bar{X}}, t) &:= \overline{\text{Grad}} \overline{\boldsymbol{\varphi}}(\mathbf{\bar{X}}, t), \quad \mathbf{\bar{V}} := \text{D}_t \overline{\boldsymbol{\varphi}}(\mathbf{\bar{X}}, t). \end{aligned} \quad (1)$$

Thereby the interface gradient and divergence operators, respectively, read

$$\begin{aligned} \overline{\text{Grad}}\{\bullet\} &:= \text{Grad}\{\bullet\} \cdot \mathbf{\bar{I}} \quad \text{and} \quad \overline{\text{Div}}\{\bullet\} := \overline{\text{Grad}}\{\bullet\} : \mathbf{\bar{I}} \\ \text{with } \mathbf{\bar{I}} &:= \mathbf{I} - \mathbf{\bar{N}} \otimes \mathbf{\bar{N}}, \end{aligned} \quad (2)$$

where $\mathbf{\bar{I}}$ and $\mathbf{I}$ denote the interface and bulk unit tensors. Their spatial counterparts are denoted $\bar{\mathbf{i}}$ and $\mathbf{i}$. Finally the bulk and interface Jacobians are denoted by $J := \det \mathbf{F} > 0$, and $\bar{J} := \overline{\det} \mathbf{\bar{F}} > 0$, respectively, with $\overline{\det}\{\bullet\}$ denoting the area determinant [51].

3. Governing equations

Equilibrium conditions in the bulk and on the interface together with associated boundary conditions are listed in Table 2 (see [25, 27, 33] for further details).

The interface energy and entropy balances are presented for the case of highly conductive interfaces.

The here considered interface model deals with (i) geometrically coherent, $[\![\boldsymbol{\varphi}]\!] = \mathbf{0}$, (ii) thermo-mechanically energetic and (iii) highly conductive interfaces. The second characteristic leads to $[\![\mathbf{P}]\!] \cdot \mathbf{\bar{N}} \neq \mathbf{0}$. The third property is characterized by allowing a jump of the normal heat flux across the interface, $[\![\mathbf{Q}]\!] \cdot \mathbf{\bar{N}} \neq 0$, while the temperature jump across the interface vanishes, that is, $[\![\Theta]\!] = 0$. The non-vanishing jump of the normal heat flux across the interface is the consequence of assuming a thermal interface. Recall that a thermal interface is non-energetic and allows heat conduction along the interface, $\overline{\mathbf{Q}} \neq \mathbf{0}$.

Furthermore, the interface undergoes in-plane degradation which affects both thermal and mechanical properties of the interface. Such effects are taken into account by introducing a reduction factor $[1 - \overline{D}_\parallel]$ in accordance with a continuum damage approach, which reduces the mechanical and thermal properties of the interface as the damage $\overline{D}_\parallel$ evolves. To avoid pathological mesh-dependency associated with local damage models, an integral-type non-localization is utilized. Note that in this work the damage variable is a function of the non-local equivalent distortion $\overline{F}_{\text{nlc}}$, which in turn depends on the interface deformation gradient $\mathbf{\bar{F}}$ and temperature $\overline{\Theta}$.

The interface Piola stress $\mathbf{\bar{P}}$ is a superficial tensor field possessing the property $\mathbf{\bar{P}} \cdot \mathbf{\bar{N}} = \mathbf{0}$. Since the interface is energetic, in the absence of $\mathbf{\bar{B}}^p$, the jump of traction across the interface equates with the negative divergence of the interface stress tensor. Therefore, the classical traction continuity across the interface no longer holds.

The arguments of the bulk and interface free energy as a Legendre transformation of the internal energy for thermo-hyperelastic behavior, are selected as

$$\Psi \equiv \Psi(\mathbf{F}, \Theta) \text{ and } \bar{\Psi} = \bar{\Psi}(\bar{\mathbf{F}}, \bar{\Theta}, \bar{D}_{\parallel}, \bar{\vartheta}), \quad (3)$$

where $0 \leq \bar{D}_{\parallel}(\bar{\mathbf{F}}, \bar{\Theta}) \leq 1$, $\bar{\Psi} = [1 - \bar{D}_{\parallel}] \bar{\Psi}^0$ and $\bar{\vartheta}$ is an internal variable which will be discussed later.

Exploiting the Coleman–Noll procedure in the bulk and on the interface yields the constitutive relations

$$\begin{aligned} \mathbf{P} &:= \frac{\partial \Psi}{\partial \mathbf{F}}, \quad \Xi := -\frac{\partial \Psi}{\partial \Theta} \quad \text{and} \\ \bar{\mathbf{P}} &:= \frac{\partial \bar{\Psi}}{\partial \bar{\mathbf{F}}} = [1 - \bar{D}_{\parallel}] \bar{\mathbf{P}}_0, \quad \bar{\Xi} := -\frac{\partial \bar{\Psi}}{\partial \bar{\Theta}} = [1 - \bar{D}_{\parallel}] \bar{\Xi}^0. \end{aligned} \quad (4)$$

Furthermore, the heat flux vectors in the bulk and on the interface take the forms

$$\begin{aligned} \mathbf{Q} &= -J \mathbf{F}^{-1} \cdot \mathbf{k} \cdot \mathbf{F}^{-t} \cdot \text{Grad} \Theta \quad \text{and} \\ \bar{\mathbf{Q}} &= -\bar{J} \bar{\mathbf{F}}^{-1} \cdot [1 - \bar{D}_{\parallel}] \bar{\mathbf{k}}_0 \cdot \bar{\mathbf{F}}^{-t} \cdot \text{Grad} \bar{\Theta}, \end{aligned} \quad (5)$$

where $\mathbf{k}$ and $\bar{\mathbf{k}}_0$ denotes the bulk and interface positive (semi-)definite thermal conductivity tensors. In the case of thermally isotropic materials in the spatial configuration, $\mathbf{k} = k \mathbf{i}$ and $\bar{\mathbf{k}}_0 = \bar{k}_0 \bar{\mathbf{i}}$, where the scalars $k \geq 0$ and $\bar{k}_0 \geq 0$ are the thermal conductivity coefficients in the bulk and on the interface, respectively. By combining the thermo-hyperelastic constitutive relations (4) and the local energy balance equations (see Table 2) the temperature evolution equations in the bulk and on the interface are obtained as

$$c_F D_t \Theta = -\text{Div} \mathbf{Q} + \Theta \partial_{\Theta} \mathbf{P} : D_t \mathbf{F} + \bar{Q}^p \quad \text{and}, \quad (6)$$

$$\bar{c}_{\bar{F}} D_t \bar{\Theta} = -\bar{\text{Div}} \bar{\mathbf{Q}} + \bar{\Theta} \partial_{\bar{\Theta}} \bar{\mathbf{P}} : D_t \bar{\mathbf{F}} + \bar{Q}^p - \llbracket \mathbf{Q} \rrbracket \cdot \bar{\mathbf{N}}, \quad (7)$$

$$\text{with } c_F := -\Theta \frac{\partial^2 \Psi}{\partial \Theta^2} \quad \text{and} \quad \bar{c}_{\bar{F}} := -[1 - \bar{D}_{\parallel}] \bar{\Theta} \frac{\partial^2 \bar{\Psi}^0}{\partial \bar{\Theta}^2},$$

where c_F and $\bar{c}_{\bar{F}} = [1 - \bar{D}_{\parallel}] \bar{c}_{\bar{F}}^0$ denote the heat capacity coefficients in the bulk and on the interface. Note that $\bar{c}_{\bar{F}}^0$ is the interface heat capacity coefficient associated with the undamaged (virgin) state of the interface material.

Next, we briefly present a rate-independent isotropic non-local damage formulation for large deformations on the interface. Physically, the nucleation, growth and coalescence of micro-voids/cracks are the reasons for the degradation of material properties. Considering continuum mechanics, the process resulting in material degradation can be modeled by introducing an internal variable denoted by $\bar{D}_{\parallel}$. To proceed a Helmholtz energy¹ is considered for the interface containing the following arguments [52]

$$\bar{\Psi}(\bar{\mathbf{F}}, \bar{\Theta}, \bar{D}, \bar{\vartheta}) = [1 - \bar{D}] \bar{\Psi}^0(\bar{\mathbf{F}}, \bar{\Theta}) + \int_0^{\bar{\vartheta}} \bar{\mathcal{H}}(\bar{\vartheta}^*) d\bar{\vartheta}^*, \quad (8)$$

where $\bar{\mathcal{H}}(\bar{\vartheta})$ denotes a monotonically increasing function depending on the internal variable $\bar{\vartheta}$.

In order to find the internal interface dissipation $\bar{\mathcal{D}}_{\text{int}}$, we differentiate equation (8) with respect to time leading to

$$\dot{\bar{\Psi}} = [1 - \bar{D}_{\parallel}] \left[\partial_{\bar{\mathbf{F}}} \bar{\Psi}^0 : \dot{\bar{\mathbf{F}}} + \partial_{\bar{\Theta}} \bar{\Psi}^0 \dot{\bar{\Theta}} \right] - \dot{\bar{D}}_{\parallel} \bar{\Psi}^0 + \bar{\mathcal{H}}(\bar{\vartheta}) \dot{\bar{\vartheta}}. \quad (9)$$

By particularizing the Clausius-Plank inequality and using equation (9) one finds

$$\begin{aligned} \bar{\mathcal{D}}_{\text{int}} &= \left[\bar{\mathbf{P}} - [1 - \bar{D}_{\parallel}] \frac{\partial \bar{\Psi}^0(\bar{\mathbf{F}}, \bar{\Theta})}{\partial \bar{\mathbf{F}}} \right] : \dot{\bar{\mathbf{F}}} \\ &\quad - \left[\bar{\Xi} + [1 - \bar{D}_{\parallel}] \frac{\partial \bar{\Psi}^0(\bar{\mathbf{F}}, \bar{\Theta})}{\partial \bar{\Theta}} \right] \dot{\bar{\Theta}} \\ &\quad + \bar{\Psi}^0(\bar{\mathbf{F}}, \bar{\Theta}) \dot{\bar{D}}_{\parallel} - \bar{\mathcal{H}}(\bar{\vartheta}) \dot{\bar{\vartheta}} \geq 0. \end{aligned} \quad (10)$$

Therefore, the interface reduced dissipation $\overline{\mathcal{D}}_{\text{red}}$ is expressed as

$$\overline{\mathcal{D}}_{\text{red}} = \overline{Y} \dot{\overline{D}}_{\parallel} - \mathcal{H}(\overline{\vartheta}) \dot{\overline{\vartheta}} \geq 0 \text{ with } \overline{Y} = -\frac{\partial \overline{\Psi}}{\partial \overline{D}_{\parallel}}, \quad (11)$$

where the quantity $\overline{Y} = \overline{\Psi}^0(\overline{F}, \overline{\vartheta})$, driving the damage evolution, is the thermodynamic force conjugate to the interface damage variable $\overline{D}_{\parallel}$. Next a damage condition $\overline{\Upsilon}$ is motivated as [53]

$$\overline{\Upsilon}(\overline{Y}, \overline{\mathcal{H}}) = \overline{v}(\overline{Y}) - \overline{\mathcal{H}}(\overline{\vartheta}) \leq 0, \quad (12)$$

with $\overline{v}$ being a monotonically increasing function. The Kuhn–Tucker conditions then read

$$\overline{\Upsilon}(\overline{Y}, \overline{\mathcal{H}}) \leq 0, \quad \overline{\lambda}^* \geq 0, \quad \overline{\lambda}^* \overline{\Upsilon}(\overline{Y}, \overline{\mathcal{H}}) = 0, \quad (13)$$

with $\overline{\lambda}^*$ being the consistency parameter. Now by choosing $\overline{v}(\bullet) = \overline{\mathcal{H}}(\bullet)$, and defining the change of variables $\overline{F}_{\text{max}} := f(\overline{\vartheta})$ and $\overline{F}_{\text{nloc}} := f(\overline{Y})$ and assuming f to be a monotonically increasing function with the property $f(0) = 0$, an alternative damage condition to equation (12) takes the form [53]

$$\begin{aligned} \overline{\phi}(\overline{F}_{\text{nloc}}, \overline{F}_{\text{max}}) &= \overline{F}_{\text{nloc}} - \overline{F}_{\text{max}} \leq 0 \\ \text{with } \overline{F}_{\text{nloc}}(\overline{\mathbf{x}}_r) &= \int_{\mathcal{I}_0} \overline{\omega}(\overline{\mathbf{x}}_r, \overline{\mathbf{x}}_s) \overline{F}_{\text{loc}}(\overline{\mathbf{x}}_s) dA, \end{aligned} \quad (14)$$

where $\overline{F}_{\text{max}}(t) = \max_{s \in [0, t]} \{\overline{F}_0, \overline{F}_{\text{nloc}}|_s\}$, $\overline{F}_{\text{loc}} := \sqrt{2\overline{Y}/\overline{E}}$ is the local equivalent distortion, $\overline{F}_0$ is the damage threshold and $\overline{E}$ is the interface Young's modulus. Note that the damage variable is eventually simply a function of $\overline{F}_{\text{max}}$, that is, $\overline{D}_{\parallel} = \overline{\mathcal{D}}_{\parallel}(\overline{F}_{\text{max}})$. The evolution of damage occurs when $\overline{\phi} = 0$, which characterizes the damage surface. In equation (14)₂, $\overline{\omega}(\overline{\mathbf{x}}_r, \overline{\mathbf{x}}_s)$ is a given non-local weight function depending on the geodesic distance² $\overline{r} = \|\overline{\mathbf{x}}_r - \overline{\mathbf{x}}_s\|_{\mathcal{I}}$ between the source point $\overline{\mathbf{x}}_s$ and the receiver point $\overline{\mathbf{x}}_r$. Note that in equation (14)₂, the integral extends over the lower-dimensional manifold $\mathcal{I}_0$. In order to resolve the issue of alteration of a uniform field in the vicinity of the boundary, the weight function is scaled leading to

$$\begin{aligned} \overline{\omega}(\overline{\mathbf{x}}_r, \overline{\mathbf{x}}_s) &= \frac{\overline{\omega}_0(\overline{r})}{\int_{\mathcal{I}_0} \overline{\omega}_0(\overline{r}) dA} \quad \text{with} \\ \overline{\omega}_0(\overline{r}) &= \begin{cases} \left[1 - \frac{\overline{r}^2}{\overline{R}^2}\right]^2 & \text{if } |\overline{r}| \leq \overline{R}, \\ 0 & \text{if } |\overline{r}| \geq \overline{R}, \end{cases} \end{aligned} \quad (15)$$

where $\overline{\omega}_0(\overline{r})$ is a non-negative and monotonically decreasing (for $\overline{r} \geq 0$) piecewise polynomial bell-shaped function. The interface interaction radius is denoted by $\overline{R}$. Finally, a smooth function to relate the damage variable $\overline{D} = \overline{\mathcal{D}}(\overline{F}_{\text{max}})$ to the history variable $\overline{F}_{\text{max}}$ with an exponential softening law is chosen as follows

$$\overline{\mathcal{D}}(\overline{F}_{\text{max}}) = \begin{cases} 0 & \text{if } \overline{F}_{\text{max}} \leq \overline{F}_0 \\ 1 - \frac{\overline{F}_0}{\overline{F}_{\text{max}}} \exp\left(-\frac{\overline{F}_{\text{max}} - \overline{F}_0}{\overline{F}_f - \overline{F}_0}\right) & \text{if } \overline{F}_{\text{max}} \geq \overline{F}_0, \end{cases} \quad (16)$$

where $\overline{F}_f$ affects the ductility of the response. An illustration of such response is depicted in Figure 3(a).

4. Computational framework

The purpose of this section is to establish a numerical framework that encompasses thermo-hyperelasticity combined with a non-local damage model on the highly conductive interface. Deriving the weak form, together with the temporal and spatial discretizations will be presented next.

Figure 3. Stress versus equivalent distortion with exponential softening on the interface (a). Thermal and geometrical descriptions of the interface and illustration of the thermal and geometrical degrees of freedom (b). The parameters $\bar{F}_0$ and $\bar{F}_f$ are the interface critical equivalent distortion and ductility response. The global stiffness matrix consists of contributions from the interface and the bulk. The interface temperature under the highly conductive assumption is the average of temperatures across the interface.

4.1. Weak form

To derive the mechanical weak form, the localized force balance equations in the bulk and on the interface given in Table 2 are tested (from the left) with vector valued test functions $\delta\boldsymbol{\varphi} \in \mathcal{H}^1(\mathcal{B}_0)$ and $\delta\bar{\boldsymbol{\varphi}} \in \mathcal{H}^1(\mathcal{I}_0)$, respectively. The result is then integrated over the corresponding domains in the material configuration. Using the bulk and interface divergence theorems and the superficiality properties of the interface Piola stress, the weak form of the balance of linear momentum is

$$\begin{aligned}
 & \int_{\mathcal{B}_0} \mathbf{P} : \text{Grad} \delta\boldsymbol{\varphi} dV + \int_{\mathcal{I}_0} [1 - \bar{D}_{\parallel}] \bar{\mathbf{P}}_0 : \overline{\text{Grad}} \delta\bar{\boldsymbol{\varphi}} dA \\
 & - \int_{\mathcal{B}_0} \delta\boldsymbol{\varphi} \cdot \mathbf{B}^p dV - \int_{\mathcal{I}_0} \delta\bar{\boldsymbol{\varphi}} \cdot \bar{\mathbf{B}}^p dA \\
 & - \int_{\partial\mathcal{B}_0^N} \delta\boldsymbol{\varphi} \cdot \widehat{\mathbf{B}}_N^p dA - \int_{\partial\mathcal{I}_0^N} \delta\bar{\boldsymbol{\varphi}} \cdot \widetilde{\mathbf{B}}_N^p dL = 0, \\
 & \forall \delta\boldsymbol{\varphi} \in \mathcal{H}^1(\mathcal{B}_0), \quad \forall \delta\bar{\boldsymbol{\varphi}} \in \mathcal{H}^1(\mathcal{I}_0) \quad \text{with} \\
 & \delta\bar{\boldsymbol{\varphi}} = \{\delta\boldsymbol{\varphi}\}|_{\mathcal{I}_0} \quad \text{and} \quad [[\delta\boldsymbol{\varphi}]] = \mathbf{0}.
 \end{aligned} \tag{17}$$

Analogously the thermal weak form is derived first by testing the local temperature evolutions equation (6) and (7) in the bulk and on the interface with the scalar-valued test function $\delta\Theta \in \mathcal{H}_0^1(\mathcal{B}_0)$ and $\delta\bar{\Theta} \in \mathcal{H}_0^1(\mathcal{I}_0)$, respectively. The result is then integrated over the corresponding domains in the material configuration. The global weak form of the temperature evolution equation reads

$$\begin{aligned}
 & \int_{\mathcal{B}_0} \mathbf{Q} \cdot \text{Grad} \delta\Theta - \delta\Theta c_F D_t \Theta + \delta\Theta \Theta \partial_{\Theta} \mathbf{P} : D_t \mathbf{F} dV \\
 & + \int_{\mathcal{B}_0} \delta\Theta Q^p dV + \int_{\partial\mathcal{B}_0^N} \delta\Theta \widehat{Q}_N^p dA + \int_{\mathcal{I}_0} \delta\bar{\Theta} \bar{Q}^p dA \\
 & + \int_{\mathcal{I}_0} [1 - \bar{D}_{\parallel}] \bar{\mathbf{Q}}_0 \cdot \overline{\text{Grad}} \delta\bar{\Theta} - \delta\bar{\Theta} [1 - \bar{D}_{\parallel}] \bar{c}_F^0 D_t \bar{\Theta} dA \\
 & + \int_{\mathcal{I}_0} \delta\bar{\Theta} \bar{\Theta} \left[[1 - \bar{D}_{\parallel}] \partial_{\bar{\Theta}} \bar{\mathbf{P}}_0 - \partial_{\bar{\Theta}} \bar{\mathcal{D}}_{\parallel} \bar{\mathbf{P}}_0 \right] : D_t \bar{\mathbf{F}} dA = 0, \\
 & \forall \delta\Theta \in \mathcal{H}_0^1(\mathcal{B}_0), \quad \forall \delta\bar{\Theta} \in \mathcal{H}_0^1(\mathcal{I}_0) \quad \text{with} \\
 & \delta\bar{\Theta} = \{\delta\Theta\}|_{\mathcal{I}_0} \quad \text{and} \quad [[\delta\Theta]] = 0,
 \end{aligned} \tag{18}$$

where $\bar{\mathbf{Q}}_0 = -\bar{\mathbf{J}}\bar{\mathbf{F}}^{-1} \cdot \bar{\mathbf{k}}_0 \cdot \bar{\mathbf{F}}^{-t} \cdot \text{Grad } \bar{\Theta}$, is the undamaged heat conduction along the interface.

4.2. Finite element implementation

In order to apply the finite element method to the present problem, the weak forms equation (17) and (18) are discretized. The discretization is carried out first in time using a classical Euler–backward integration scheme and subsequently in space by means of the finite element method.

The time interval $\mathcal{T}$ is subdivided into a set of intervals $\Delta t := t_{\tau+1} - t_\tau$ with

$$\mathcal{T} = \bigcup_{\tau=0}^{n_{ts}} [t_\tau, t_{\tau+1}] , \quad (19)$$

where n_{ts} denotes the number of time steps.

Next the spatial discretization of the solution domain is performed using the Bubnov–Galerkin finite element method. The geometry and temperature of the bulk and interface are approximated as a function of the natural coordinates $\xi \in [-1, 1]^3$ and $\bar{\xi} \in [-1, 1]^2$ assigned to the bulk and the interface, respectively, using standard interpolations according to the isoparametric concepts as follows:

$$\begin{aligned} \mathbf{X} |_{\mathcal{B}_0^\beta} &\approx \mathbf{X}^h(\xi) = \sum_{i=1}^{n_{nB}} N^i(\xi) \mathbf{X}^i , \\ \bar{\mathbf{X}} |_{\mathcal{I}_0^\gamma} &\approx \bar{\mathbf{X}}^h(\bar{\xi}) = \sum_{i=1}^{n_{nI}} \bar{N}^i(\bar{\xi}) \bar{\mathbf{X}}^i , \\ \boldsymbol{\varphi} |_{\mathcal{B}_0^\beta} &\approx \boldsymbol{\varphi}^h(\xi) = \sum_{i=1}^{n_{nB}} N^i(\xi) \boldsymbol{\varphi}^i , \\ \bar{\boldsymbol{\varphi}} |_{\mathcal{I}_0^\gamma} &\approx \bar{\boldsymbol{\varphi}}^h(\bar{\xi}) = \sum_{i=1}^{n_{nI}} \bar{N}^i(\bar{\xi}) \bar{\boldsymbol{\varphi}}^i , \\ \Theta |_{\mathcal{B}_0^\beta} &\approx \Theta^h(\xi) = \sum_{i=1}^{n_{nB}} N^i(\xi) \Theta^i , \\ \bar{\Theta} |_{\mathcal{I}_0^\gamma} &\approx \bar{\Theta}^h(\bar{\xi}) = \sum_{i=1}^{n_{nI}} \bar{N}^i(\bar{\xi}) \bar{\Theta}^i , \end{aligned} \quad (20)$$

where $\mathcal{B}_0^\beta$ and $\mathcal{I}_0^\gamma$ are the β^{th} and γ^{th} element in the bulk and on the interface, respectively. The shape functions of the bulk and interface elements at a local node i are denoted by N^i and $\bar{N}^i$, respectively. Every bulk and interface element consists of n_{nB} and n_{nI} nodes, respectively.

The interface is a two-dimensional manifold embedded in the three-dimensional space and therefore can be described by two surface coordinates. The corresponding tangent vectors to the coordinate lines, that is, the covariant interface basis vectors, are obtained by taking the derivative of the position vector on the interface with respect to the interface coordinates. The covariant basis vectors then allow the normal to the interface to be computed (see [54] for further details).

Now, the fully discrete (spatially and temporally) form of mechanical and thermal residuals associated with the global node I are defined by³

$$\begin{aligned} [\text{tot} \mathbf{R}_\varphi^I]_{\tau+1} &= \int_{\mathcal{B}_0} \mathbf{P}_{\tau+1} \cdot \text{Grad} N^I dV \\ &\quad + \int_{\mathcal{I}_0} [1 - \bar{D}_\parallel] \bar{\mathbf{P}}_0 \cdot \text{Grad } \bar{N}^I dA \\ &\quad - \int_{\mathcal{B}_0} N^I \mathbf{B}_{\tau+1}^p dV - \int_{\mathcal{I}_0} \bar{N}^I \bar{\mathbf{B}}_{\tau+1}^p dA , \end{aligned} \quad (21)$$

and

$$\begin{aligned}
[\text{tot} \mathbf{R}_{\Theta}^I]_{\tau+1} = & - \int_{\mathcal{B}_0} \mathbf{Q}_{\tau+1} \cdot \text{Grad} N^I dV - \\
& \int_{\mathcal{I}_0} [1 - D_{\parallel}] \bar{\mathbf{Q}}_0]_{\tau+1} \cdot \overline{\text{Grad}} \bar{N}^I dA \\
& - \int_{\mathcal{B}_0} \Theta_{\tau+1} [\partial_{\Theta} \mathbf{P}]_{\tau+1} : \frac{[\mathbf{F}_{\tau+1} - \mathbf{F}_{\tau}]}{\Delta t} N^I dV \\
& + \int_{\mathcal{B}_0} c_F \frac{[\Theta_{\tau+1} - \Theta_{\tau}]}{\Delta t} N^I dV \\
& - \int_{\mathcal{I}_0} \bar{\Theta}_{\tau+1} [\partial_{\bar{\Theta}} ([1 - \bar{D}_{\parallel}] \bar{\mathbf{P}}_0)]_{\tau+1} : \frac{[\bar{\mathbf{F}}_{\tau+1} - \bar{\mathbf{F}}_{\tau}]}{\Delta t} \bar{N}^I dA \\
& + \int_{\mathcal{I}_0} [1 - \bar{D}_{\parallel}] \bar{c}_{\bar{F}}^0 \frac{1}{\Delta t} [\bar{\Theta}_{\tau+1} - \bar{\Theta}_{\tau}] \bar{N}^I dA \\
& - \int_{\mathcal{B}_0} \mathcal{Q}_{\tau+1}^p N^I dV - \int_{\mathcal{I}_0} \bar{\mathcal{Q}}_{\tau+1}^p \bar{N}^I dA.
\end{aligned} \tag{22}$$

Note that mechanical and thermal residuals are composed of contributions from both the bulk and interface. The global mechanical and thermal residual vectors take the form

$$\begin{aligned}
\text{tot} \mathbf{R} = \begin{bmatrix} \text{tot} \mathbf{R}_{\varphi} \\ \text{tot} \mathbf{R}_{\Theta} \end{bmatrix} \quad \text{with} \quad \text{tot} \mathbf{R}_{\varphi} = \begin{bmatrix} \mathbf{R}_{\varphi}^1 \\ \vdots \\ \mathbf{R}_{\varphi}^I \\ \vdots \\ \mathbf{R}_{\varphi}^{n_{\text{Bn}}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{R}}_{\varphi}^1 \\ \vdots \\ \bar{\mathbf{R}}_{\varphi}^I \\ \vdots \\ \bar{\mathbf{R}}_{\varphi}^{n_{\text{In}}} \end{bmatrix} \\
\text{and} \quad \text{tot} \mathbf{R}_{\Theta} = \begin{bmatrix} \mathbf{R}_{\Theta}^1 \\ \vdots \\ \mathbf{R}_{\Theta}^I \\ \vdots \\ \mathbf{R}_{\Theta}^{n_{\text{Bn}}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{R}}_{\Theta}^1 \\ \vdots \\ \bar{\mathbf{R}}_{\Theta}^I \\ \vdots \\ \bar{\mathbf{R}}_{\Theta}^{n_{\text{In}}} \end{bmatrix},
\end{aligned} \tag{23}$$

where n_{Bn} and n_{In} denote the number of bulk and interface nodes. The summation operator implies the (conventional) residual assembly of the finite element method.

The fully-discrete coupled non-linear system of governing equations can be stated as follows:

$$\text{tot} \mathbf{R}(\text{tot} \mathbf{d}) \stackrel{!}{=} \mathbf{0} \quad \text{with} \quad \text{tot} \mathbf{d} = \begin{bmatrix} \mathbf{d}_{\varphi} \\ \mathbf{d}_{\Theta} \end{bmatrix} \tag{24}$$

where $\text{tot} \mathbf{d}$ is the unknown global vector of spatial coordinates $\mathbf{d}_{\varphi}$ and temperature $\mathbf{d}_{\Theta}$. To solve (24)₁, a Newton–Raphson scheme is utilized. Thus, the consistent linearization of the resulting system of equations yields the total (algorithmic) tangent stiffness matrix for every pair of global nodes I and J , as

$$\begin{aligned}
\text{tot} \mathbf{K}^{IJ} &:= \frac{\partial \text{tot} \mathbf{R}^I}{\partial \text{tot} \mathbf{d}^J} \quad \text{where} \\
\text{tot} \mathbf{K}^{IJ} &= \begin{bmatrix} \mathbf{K}_{\varphi\varphi}^{IJ} & \mathbf{K}_{\varphi\Theta}^{IJ} \\ \mathbf{K}_{\Theta\varphi}^{IJ} & \mathbf{K}_{\Theta\Theta}^{IJ} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{K}}_{\varphi\varphi}^{IJ} & \bar{\mathbf{K}}_{\varphi\Theta}^{IJ} \\ \bar{\mathbf{K}}_{\Theta\varphi}^{IJ} & \bar{\mathbf{K}}_{\Theta\Theta}^{IJ} \end{bmatrix}.
\end{aligned} \tag{25}$$

Note that ${}^{\text{tot}}\mathbf{K}$ is also decomposed into contributions from the bulk and the interface (see Figure 3(b)). The bulk contributions to the total stiffness matrix are as follows:

$$\begin{aligned}
 \mathbf{K}_{\varphi\varphi}^{\text{II}} &= \int_{\mathcal{B}_0} \text{Grad } N^I \cdot [\partial_{\mathbf{F}} \mathbf{P}] \cdot \text{Grad } N^J \, dV, \\
 \mathbf{K}_{\varphi\Theta}^{\text{II}} &= \int_{\mathcal{B}_0} N^J [\partial_{\Theta} \mathbf{P}] \cdot \text{Grad } N^I \, dV, \\
 \mathbf{K}_{\Theta\varphi}^{\text{II}} &= \int_{\mathcal{B}_0} -N^I \left[\frac{\partial(\Theta \partial_{\Theta} \mathbf{P} : \mathbf{D}_t \mathbf{F})}{\partial \mathbf{F}} \right] \cdot \text{Grad } N^J \, dV \\
 &\quad - \int_{\mathcal{B}_0} \text{Grad } N^I \cdot [\partial_{\mathbf{F}} \mathbf{Q}] \cdot \text{Grad } N^J \, dV, \\
 \mathbf{K}_{\Theta\Theta}^{\text{II}} &= \int_{\mathcal{B}_0} -\text{Grad } N^I \cdot \left[\frac{\partial \mathbf{Q}}{\partial \text{Grad } \Theta} \right] \cdot \text{Grad } N^J \, dV \\
 &\quad + \int_{\mathcal{B}_0} N^I \left[\frac{c_{\mathbf{F}}}{\Delta t} - \left[\frac{\partial(\Theta \partial_{\Theta} \mathbf{P} : \mathbf{D}_t \mathbf{F})}{\partial \Theta} \right] \right] N^J \, dV.
 \end{aligned} \tag{26}$$

The interface contributions to the total stiffness matrix consist of local ${}^{\text{loc}}\overline{\mathbf{K}}$ and non-local ${}^{\text{nloc}}\overline{\mathbf{K}}$ contributions. The local contributions ${}^{\text{loc}}\overline{\mathbf{K}}$ are as follows:

$$\begin{aligned}
 {}^{\text{loc}}\overline{\mathbf{K}}_{\varphi\varphi}^{\text{II}} &= \int_{\mathcal{I}_0} \overline{\text{Grad}} \, \overline{N}^I \cdot [1 - \overline{D}_{\parallel}] \partial_{\overline{\mathbf{F}}} \overline{\mathbf{P}}_0 \cdot \overline{\text{Grad}} \, \overline{N}^J \, dA_{\text{r}}, \\
 {}^{\text{loc}}\overline{\mathbf{K}}_{\varphi\Theta}^{\text{II}} &= \int_{\mathcal{I}_0} \overline{N}^J [1 - \overline{D}_{\parallel}] \partial_{\overline{\Theta}} \overline{\mathbf{P}}_0 \cdot \overline{\text{Grad}} \, \overline{N}^I \, dA_{\text{r}}, \\
 {}^{\text{loc}}\overline{\mathbf{K}}_{\Theta\varphi}^{\text{II}} &= \int_{\mathcal{I}_0} -\overline{\text{Grad}} \, \overline{N}^I \cdot [1 - \overline{D}_{\parallel}] \left[\partial_{\overline{\mathbf{F}}} \overline{\mathbf{Q}}_0 \right] \cdot \overline{\text{Grad}} \, \overline{N}^J \, dA_{\text{r}} \\
 &\quad - \int_{\mathcal{I}_0} \overline{N}^I [1 - \overline{D}_{\parallel}] \left[\frac{\partial(\overline{\Theta} \partial_{\overline{\Theta}} \overline{\mathbf{P}}_0 : \mathbf{D}_t \overline{\mathbf{F}})}{\partial \overline{\mathbf{F}}} \right] \cdot \overline{\text{Grad}} \, \overline{N}^J \, dA_{\text{r}} \\
 &\quad - \int_{\mathcal{I}_0} \overline{N}^I [-\partial_{\overline{\Theta}} \overline{\mathcal{D}}_{\parallel}] \left[\frac{\partial(\overline{\Theta} \overline{\mathbf{P}}_0 : \mathbf{D}_t \overline{\mathbf{F}})}{\partial \overline{\mathbf{F}}} \right] \cdot \overline{\text{Grad}} \, \overline{N}^J \, dA_{\text{r}}, \\
 {}^{\text{loc}}\overline{\mathbf{K}}_{\Theta\Theta}^{\text{II}} &= \int_{\mathcal{I}_0} -\overline{N}^I [-\partial_{\overline{\Theta}} \overline{\mathcal{D}}_{\parallel}] \left[\frac{\partial(\overline{\Theta} \overline{\mathbf{P}}_0 : \mathbf{D}_t \overline{\mathbf{F}})}{\partial \overline{\Theta}} \right] \overline{N}^J \, dA_{\text{r}} \\
 &\quad - \int_{\mathcal{I}_0} \overline{\text{Grad}} \, \overline{N}^I \cdot [1 - \overline{D}_{\parallel}] \left[\frac{\partial \overline{\mathbf{Q}}_0}{\partial \text{Grad } \overline{\Theta}} \right] \cdot \overline{\text{Grad}} \, \overline{N}^J \, dA_{\text{r}} \\
 &\quad + \int_{\mathcal{I}_0} \overline{N}^I [1 - \overline{D}_{\parallel}] \left[\frac{c_{\overline{\mathbf{F}}}^0}{\Delta t} - \left[\frac{\partial(\overline{\Theta} \partial_{\overline{\Theta}} \overline{\mathbf{P}}_0 : \mathbf{D}_t \overline{\mathbf{F}})}{\partial \overline{\Theta}} \right] \right] \overline{N}^J \, dA_{\text{r}} \\
 &\quad - \int_{\mathcal{I}_0} \overline{N}^I [-\partial_{\overline{\Theta}} \overline{\mathcal{D}}_{\parallel}] \left[\frac{\partial(\overline{\Theta} \overline{\mathbf{P}}_0 : \mathbf{D}_t \overline{\mathbf{F}})}{\partial \overline{\Theta}} \right] \overline{N}^J \, dA_{\text{r}},
 \end{aligned} \tag{27}$$

where $dA_r = dA(\bar{\mathbf{x}}_r)$. Note that all the above integrands are evaluated at node $\bar{\mathbf{x}}_r$. Using equation (14)_{2,3}, $\partial_{\bar{\Theta}} \bar{\mathcal{D}}_{\parallel}$ is computed as follows:

$$\begin{aligned}\partial_{\bar{\Theta}} \bar{\mathcal{D}}_{\parallel}(\bar{\mathbf{x}}_r) &= \bar{\mathcal{D}}'_{\parallel} \partial_{\bar{\Theta}} \bar{F}_{\text{nloc}} \implies \\ \partial_{\bar{\Theta}} \bar{F}_{\text{nloc}}(\bar{\mathbf{x}}_r) &= \int_{\mathcal{I}_0} \omega(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s) \partial_{\bar{\Theta}} \bar{F}_{\text{loc}}(\bar{\mathbf{x}}_s) dA_s \\ &= \int_{\mathcal{I}_0} \omega(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s) \frac{1}{EF_{\text{loc}}(\bar{\mathbf{x}}_s)} \partial_{\bar{\Theta}} \bar{\Psi}^0(\bar{\mathbf{x}}_s) dA_s, \end{aligned} \quad (28)$$

with $\bar{\mathcal{D}}'_{\parallel} = \partial_{\bar{F}_{\text{nloc}}} \bar{\mathcal{D}}_{\parallel}$. To facilitate the derivation of the non-local corrections to the interface stiffness matrix we calculate the first and second derivatives of damage variable with respect to the nodal coordinates and temperature $\bar{\varphi}^J$ and $\bar{\Theta}^J$, at an arbitrary point $\bar{\mathbf{x}}_r$, on the interface. Using equation (14)_{2,3} we obtain

$$\begin{aligned}\partial_{\bar{\Theta}^J} \bar{\mathcal{D}}_{\parallel}(\bar{\mathbf{x}}_r) &= \bar{\mathcal{D}}'_{\parallel} \partial_{\bar{\Theta}^J} \bar{F}_{\text{nloc}} \implies \\ \partial_{\bar{\Theta}^J} \bar{F}_{\text{nloc}}(\bar{\mathbf{x}}_r) &= \int_{\mathcal{I}_0} \omega(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s) \partial_{\bar{\Theta}^J} \bar{F}_{\text{loc}}(\bar{\mathbf{x}}_s) dA_s \\ &= \int_{\mathcal{I}_0} \omega(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s) \frac{1}{EF_{\text{loc}}(\bar{\mathbf{x}}_s)} \partial_{\bar{\Theta}} \bar{\Psi}^0(\bar{\mathbf{x}}_s) \bar{N}^J(\bar{\mathbf{x}}_s) dA_s, \end{aligned} \quad (29)$$

$$\begin{aligned}\partial_{\bar{\varphi}^J} \bar{\mathcal{D}}_{\parallel}(\bar{\mathbf{x}}_r) &= \bar{\mathcal{D}}'_{\parallel} \partial_{\bar{\varphi}^J} \bar{F}_{\text{nloc}} \implies \\ \partial_{\bar{\varphi}^J} \bar{F}_{\text{nloc}}(\bar{\mathbf{x}}_r) &= \int_{\mathcal{I}_0} \omega(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s) \partial_{\bar{\varphi}^J} \bar{F}_{\text{loc}}(\bar{\mathbf{x}}_s) dA_s \\ &= \int_{\mathcal{I}_0} \omega(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s) \left[\frac{1}{EF_{\text{loc}}} \partial_{\bar{\varphi}^J} \bar{\Psi}^0 \cdot \overline{\text{Grad}} \bar{N}^J \right]_{\bar{\mathbf{x}}_s} dA_s, \end{aligned} \quad (30)$$

$$\begin{aligned}\partial_{\bar{\varphi}^J \bar{\Theta}} \bar{\mathcal{D}}_{\parallel}(\bar{\mathbf{x}}_r) &= \frac{\partial}{\partial \bar{\varphi}^J} (\partial_{\bar{\Theta}} \bar{\mathcal{D}}_{\parallel}) = \frac{\partial}{\partial \bar{\varphi}^J} (\bar{\mathcal{D}}'_{\parallel} \partial_{\bar{\Theta}} \bar{F}_{\text{nloc}}) \\ &= \partial_{\bar{\varphi}^J} \bar{\mathcal{D}}'_{\parallel} \partial_{\bar{\Theta}} \bar{F}_{\text{nloc}} + \bar{\mathcal{D}}'_{\parallel} \partial_{\bar{\varphi}^J \bar{\Theta}} \bar{F}_{\text{nloc}} \\ &= \bar{\mathcal{D}}''_{\parallel} \partial_{\bar{\Theta}} \bar{F}_{\text{nloc}} \partial_{\bar{\varphi}^J} \bar{F}_{\text{nloc}} + \bar{\mathcal{D}}'_{\parallel} \partial_{\bar{\varphi}^J \bar{\Theta}} \bar{F}_{\text{nloc}}, \end{aligned} \quad (31)$$

$$\begin{aligned}\partial_{\bar{\Theta}^J \bar{\Theta}} \bar{\mathcal{D}}_{\parallel}(\bar{\mathbf{x}}_r) &= \frac{\partial}{\partial \bar{\Theta}^J} (\partial_{\bar{\Theta}} \bar{\mathcal{D}}_{\parallel}) = \frac{\partial}{\partial \bar{\Theta}^J} (\bar{\mathcal{D}}'_{\parallel} \partial_{\bar{\Theta}} \bar{F}_{\text{nloc}}) \\ &= \partial_{\bar{\Theta}^J} \bar{\mathcal{D}}'_{\parallel} \partial_{\bar{\Theta}} \bar{F}_{\text{nloc}} + \bar{\mathcal{D}}'_{\parallel} \partial_{\bar{\Theta}^J \bar{\Theta}} \bar{F}_{\text{nloc}} \\ &= \bar{\mathcal{D}}''_{\parallel} \partial_{\bar{\Theta}^J} \bar{F}_{\text{nloc}} \partial_{\bar{\Theta}} \bar{F}_{\text{nloc}} + \bar{\mathcal{D}}'_{\parallel} \partial_{\bar{\Theta}^J \bar{\Theta}} \bar{F}_{\text{nloc}}, \end{aligned} \quad (32)$$

where $\bar{\mathcal{D}}''_{\parallel} = \partial_{\bar{F}_{\text{nloc}}} \bar{\mathcal{D}}'_{\parallel}$, $dA_s = dA(\bar{\mathbf{x}}_s)$, and

$$\begin{aligned}\partial_{\bar{\varphi}^J \bar{\Theta}} \bar{F}_{\text{nloc}}(\bar{\mathbf{x}}_r) &= \\ \int_{\mathcal{I}_0} \omega(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s) \left[\frac{-1}{E^2 \bar{F}_{\text{loc}}^3} \partial_{\bar{\varphi}^J} \bar{\Psi}^0 \partial_{\bar{\Theta}} \bar{\Psi}^0 \cdot \overline{\text{Grad}} \bar{N}^J \right]_{\bar{\mathbf{x}}_s} dA_s \\ + \int_{\mathcal{I}_0} \omega(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s) \left[\frac{1}{EF_{\text{loc}}} \partial_{\bar{\varphi}^J \bar{\Theta}} \bar{\Psi}^0 \cdot \overline{\text{Grad}} \bar{N}^J \right]_{\bar{\mathbf{x}}_s} dA_s, \end{aligned} \quad (33)$$

$$\begin{aligned}\partial_{\bar{\Theta}^J \bar{\Theta}} \bar{F}_{\text{nloc}}(\bar{\mathbf{x}}_r) &= \\ \int_{\mathcal{I}_0} \omega(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s) \left[\frac{-1}{E^2 \bar{F}_{\text{loc}}^3} [\partial_{\bar{\Theta}} \bar{\Psi}^0]^2 \bar{N}^J \right]_{\bar{\mathbf{x}}_s} dA_s \\ + \int_{\mathcal{I}_0} \omega(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s) \left[\frac{1}{EF_{\text{loc}}} \partial_{\bar{\Theta}^J \bar{\Theta}} \bar{\Psi}^0 \bar{N}^J \right]_{\bar{\mathbf{x}}_s} dA_s. \end{aligned} \quad (34)$$

Note that the notation $[\{\bullet\}]_{\bar{x}_s}$ means that the quantity $\{\bullet\}$ is evaluated at point $\bar{x}_s$. It is also important to point out the difference between the derivative of the damage variable with respect to the nodal temperature $\bar{\Theta}^J$ and the temperature $\bar{\Theta}$ (compare equation (29) with equation (28)). Such distinction is required to achieve a consistent interface stiffness matrix. Furthermore during unloading we set $\bar{\mathcal{D}}' = \bar{\mathcal{D}}'' = 0$. Finally the non-local corrections to the interface stiffness matrix are given as

$$\begin{aligned}
{}^{\text{nloc}}\bar{\mathbf{K}}_{\varphi\varphi}^U &= \int_{\mathcal{I}_0} \overline{\text{Grad } \bar{N}^I} \cdot \bar{\mathbf{P}}_0(\bar{\mathbf{x}}_r) \otimes \left[-\partial_{\varphi^J} \bar{\mathcal{D}}_{\parallel}(\bar{\mathbf{x}}_r) \right] dA_r, \\
{}^{\text{nloc}}\bar{\mathbf{K}}_{\varphi\Theta}^U &= \int_{\mathcal{I}_0} \left[-\partial_{\bar{\Theta}^J} \bar{\mathcal{D}}_{\parallel}(\bar{\mathbf{x}}_r) \right] \bar{\mathbf{P}}_0(\bar{\mathbf{x}}_r) \cdot \overline{\text{Grad } \bar{N}^I} dA_r, \\
{}^{\text{nloc}}\bar{\mathbf{K}}_{\Theta\varphi}^U &= \int_{\mathcal{I}_0} \bar{N}^I \left[-\partial_{\bar{\Theta}^J} \bar{\mathcal{D}}_{\parallel}(\bar{\mathbf{x}}_r) \right] \bar{c}_F^0 D_t \bar{\Theta} dA_r \\
&\quad - \int_{\mathcal{I}_0} \bar{N}^I \bar{\Theta} D_t \bar{\mathbf{F}} : \left[-\partial_{\bar{\Theta}^J} \bar{\mathbf{P}}_0 \otimes \partial_{\varphi^J} \bar{\mathcal{D}}_{\parallel} \right]_{\bar{\mathbf{x}}_r} dA_r \\
&\quad - \int_{\mathcal{I}_0} \bar{N}^I \bar{\Theta} D_t \bar{\mathbf{F}} : \left[-\bar{\mathbf{P}}_0 \otimes \partial_{\bar{\Theta}^J} \bar{\mathcal{D}}_{\parallel} \right]_{\bar{\mathbf{x}}_r} dA_r \\
&\quad - \int_{\mathcal{I}_0} \overline{\text{Grad } \bar{N}^I} \cdot \bar{\mathbf{Q}}_0(\bar{\mathbf{x}}_r) \otimes \left[-\partial_{\bar{\Theta}^J} \bar{\mathcal{D}}_{\parallel}(\bar{\mathbf{x}}_r) \right] dA_r, \\
{}^{\text{nloc}}\bar{\mathbf{K}}_{\Theta\Theta}^U &= \int_{\mathcal{I}_0} -\overline{\text{Grad } \bar{N}^I} \cdot \left[-\partial_{\bar{\Theta}^J} \bar{\mathcal{D}}_{\parallel}(\bar{\mathbf{x}}_r) \right] \bar{\mathbf{Q}}_0(\bar{\mathbf{x}}_r) dA_r \\
&\quad + \int_{\mathcal{I}_0} \bar{N}^I \left[-\partial_{\bar{\Theta}^J} \bar{\mathcal{D}}_{\parallel} \bar{c}_F^0 D_t \bar{\Theta} \right]_{\bar{\mathbf{x}}_r} dA_r \\
&\quad - \int_{\mathcal{I}_0} \bar{N}^I \left[-\bar{\Theta} \partial_{\bar{\Theta}^J} \bar{\mathcal{D}}_{\parallel} \partial_{\bar{\Theta}^J} \bar{\mathbf{P}}_0 : D_t \bar{\mathbf{F}} \right]_{\bar{\mathbf{x}}_r} dA_r \\
&\quad - \int_{\mathcal{I}_0} \bar{N}^I \left[-\bar{\Theta} \partial_{\bar{\Theta}^J} \bar{\mathcal{D}}_{\parallel} \bar{\mathbf{P}}_0 : D_t \bar{\mathbf{F}} \right]_{\bar{\mathbf{x}}_r} dA_r,
\end{aligned} \tag{35}$$

where $D_t \bar{\mathbf{F}} = [\bar{\mathbf{F}}_{\tau+1} - \bar{\mathbf{F}}_{\tau}]/\Delta t$ and $D_t \bar{\Theta} = [\bar{\Theta}_{\tau+1} - \bar{\Theta}_{\tau}]/\Delta t$. Similarly, the notation $[\{\bullet\}]_{\bar{\mathbf{x}}_r}$ means that the quantity $\{\bullet\}$ is evaluated at point $\bar{\mathbf{x}}_r$.

5. Numerical examples

The objective of this section is to computationally study the theory presented in the previous sections. The thermo-mechanical response of the interface is affected by the interface in-plane degradation. It is important to point out that the solution procedure is robust and shows the asymptotically quadratic rate of convergence associated with the Newton–Raphson scheme. The computational domain is discretized using 1600 trilinear hexahedral elements.

The reversible material behavior in the bulk and on the interface is characterized by thermo-hyperelastic Helmholtz energy functions. The damage only affects the interface response by reducing the interface stiffness, heat capacity and heat conduction coefficient. The interface heat expansion coefficient is also indirectly influenced through softening of the interface stiffness. Hence, after the onset of the damage, a thermo-hyperelastic-damage response describes the interface behavior. Table 3 gathers the effective (undamaged) Helmholtz energy functions together with their corresponding derivatives both in the bulk and on the interface. The corresponding material parameters for the bulk and interface are given in Table 4. The influence of the specimen size on the overall response is also studied, captured by fixing the bulk material parameters and changing those of the interface.

Consider now the strip shown in Figure 4 where a constant displacement is prescribed at the two opposite faces. The strip is partitioned into two homogeneous domains by an interface. The width and the thickness of the strip are kept constant. The thermal boundary condition is globally adiabatic i.e. $\bar{\mathcal{Q}}^p = \bar{\mathcal{Q}}^p = 0$. The thermal initial condition is a uniformly distributed temperature $\Theta_0 = 298$ K. In order to better understand the influence

Table 3. Constitutive relations in the bulk and on the interface in the material configuration.**Bulk**

$$\begin{aligned}
\Psi &= \frac{1}{2} \lambda \ln^2 J + \frac{1}{2} \mu [\mathbf{F} : \mathbf{F} - 3 - 2 \ln J] \\
&\quad - 3\alpha\kappa [\Theta - \Theta_0] J^{-1} \ln J + c_F [\Theta - \Theta_0 - \Theta \ln(\Theta/\Theta_0)] - \Xi_0 [\Theta - \Theta_0] \\
\mathbf{P} &= \lambda \ln J \mathbf{F}^{-t} + \mu [\mathbf{F} - \mathbf{F}^{-t}] - 3\alpha\kappa J^{-1} [\Theta - \Theta_0] [1 - \ln J] \mathbf{F}^{-t} \\
\Xi &= 3\alpha\kappa J^{-1} \ln J + c_F \ln(\Theta/\Theta_0) + \Xi_0 \\
\partial_{\mathbf{F}} \mathbf{P} &= \lambda [\mathbf{F}^{-t} \otimes \mathbf{F}^{-t} + \ln J \mathbb{D}] + \mu [\mathbb{I} - \mathbb{D}] + 3\alpha\kappa [\Theta - \Theta_0] [J^{-1} [2 - \ln J] \mathbf{F}^{-t} \otimes \mathbf{F}^{-t} - J^{-1} [1 - \ln J] \mathbb{D}] \\
\partial_{\Theta} \mathbf{P} &= -3\alpha\kappa J^{-1} [1 - \ln J] \mathbf{F}^{-t} \\
\partial_{\Theta} (\Theta \partial_{\Theta} \mathbf{P} : \mathbf{D}_t \mathbf{F}) &= -3\alpha\kappa J^{-1} [1 - \ln J] \operatorname{div} \mathbf{V} \\
\partial_{\mathbf{F}} (\Theta \partial_{\Theta} \mathbf{P} : \mathbf{D}_t \mathbf{F}) &= -3\alpha\kappa \Theta [J^{-1} [\ln J - 2] \operatorname{div} \mathbf{V} \mathbf{F}^{-t} + J^{-1} [1 - \ln J] [\mathbf{D}_t \mathbf{F} : \mathbb{D} + [1/\Delta t] \mathbf{F}^{-t} : \mathbb{I}]] \\
\partial_{\mathbf{F}} \mathbf{Q} &= -Jk [\mathbf{F}^{-1} \cdot \mathbf{F}^{-t} \otimes \mathbf{F}^{-t} + \mathbb{B}] : \operatorname{Grad} \Theta, \partial_{\operatorname{Grad} \Theta} \mathbf{Q} = -Jk \mathbf{G} \\
\mathbb{D} &= \partial_{\mathbf{F}} \mathbf{F}^{-t} = -\mathbf{F}^{-t} \underline{\otimes} \mathbf{F}^{-1}, \mathbb{I} = \partial_{\mathbf{F}} \mathbf{F} = \mathbf{i} \underline{\otimes} \mathbf{I} \\
\mathbb{B} &= \partial_{\mathbf{F}} (\mathbf{F}^{-1} \cdot \mathbf{F}^{-t}) = -[\mathbf{F}^{-1} \underline{\otimes} \mathbf{F}^{-1}] \cdot \mathbf{F}^{-t} - \mathbf{F}^{-1} \cdot [\mathbf{F}^{-t} \underline{\otimes} \mathbf{F}^{-t}], \mathbf{G} = \mathbf{F}^{-1} \cdot \mathbf{F}^{-t}
\end{aligned}$$

Interface

$$\begin{aligned}
\bar{\Psi}^0 &= \frac{1}{2} \bar{\lambda} \ln^2 \bar{J} + \frac{1}{2} \bar{\mu} [\bar{\mathbf{F}} : \bar{\mathbf{F}} - 2 - 2 \ln \bar{J}] \\
&\quad - 2\bar{\alpha} \bar{\kappa} [\bar{\Theta} - \bar{\Theta}_0] \bar{J}^{-1} \ln \bar{J} + \bar{c}_{\bar{F}}^0 [\bar{\Theta} - \bar{\Theta}_0 - \bar{\Theta} \ln(\bar{\Theta}/\bar{\Theta}_0)] - \bar{\Xi}_s [\bar{\Theta} - \bar{\Theta}_0] \\
\bar{\mathbf{P}}_0 &= \partial_{\bar{\mathbf{F}}} \bar{\Psi}^0 = \bar{\lambda} \ln \bar{J} \bar{\mathbf{F}}^{-t} + \bar{\mu} [\bar{\mathbf{F}} - \bar{\mathbf{F}}^{-t}] - 2\bar{\alpha} \bar{\kappa} \bar{J}^{-1} [\bar{\Theta} - \bar{\Theta}_0] [1 - \ln \bar{J}] \bar{\mathbf{F}}^{-t} \\
\bar{\Xi}_0 &= 2\bar{\alpha} \bar{\kappa} \bar{J}^{-1} \ln \bar{J} + \bar{c}_{\bar{F}}^0 \ln(\bar{\Theta}/\bar{\Theta}_0) + \bar{\Xi}_s, \partial_{\bar{\Theta}} \bar{\Psi}^0 = -\bar{\Xi}_0 \\
\partial_{\bar{\mathbf{F}}} \bar{\mathbf{P}}_0 &= \bar{\lambda} [\bar{\mathbf{F}}^{-t} \otimes \bar{\mathbf{F}}^{-t} + \ln \bar{J} \bar{\mathbb{D}}] + \bar{\mu} [\bar{\mathbb{I}} - \bar{\mathbb{D}}] + 2\bar{\alpha} \bar{\kappa} [\bar{\Theta} - \bar{\Theta}_0] [\bar{J}^{-1} [2 - \ln \bar{J}] \bar{\mathbf{F}}^{-t} \otimes \bar{\mathbf{F}}^{-t} - \bar{J}^{-1} [1 - \ln \bar{J}] \bar{\mathbb{D}}] \\
\partial_{\bar{\Theta}} \bar{\mathbf{P}}_0 &= \partial_{\bar{\mathbf{F}} \bar{\Theta}} \bar{\Psi}^0 = -2\bar{\alpha} \bar{\kappa} \bar{J}^{-1} [1 - \ln \bar{J}] \bar{\mathbf{F}}^{-t}, \partial_{\bar{\Theta} \bar{\Theta}} \bar{\Psi}^0 = -\bar{c}_{\bar{F}}^0 \bar{\Theta}^{-1} \\
\partial_{\bar{\Theta}} (\bar{\Theta} \partial_{\bar{\Theta}} \bar{\mathbf{P}}_0 : \mathbf{D}_t \bar{\mathbf{F}}) &= -2\bar{\alpha} \bar{\kappa} \bar{J}^{-1} [1 - \ln \bar{J}] \operatorname{div} \bar{\mathbf{V}} \\
\partial_{\bar{\mathbf{F}}} (\bar{\Theta} \partial_{\bar{\Theta}} \bar{\mathbf{P}}_0 : \mathbf{D}_t \bar{\mathbf{F}}) &= -2\bar{\alpha} \bar{\kappa} \bar{\Theta} [\bar{J}^{-1} [\ln \bar{J} - 2] \operatorname{div} \bar{\mathbf{V}} \bar{\mathbf{F}}^{-t} + \bar{J}^{-1} [1 - \ln \bar{J}] [\mathbf{D}_t \bar{\mathbf{F}} : \bar{\mathbb{D}} + [1/\Delta t] \bar{\mathbf{F}}^{-t} : \bar{\mathbb{I}}]] \\
\partial_{\bar{\mathbf{F}}} \bar{\mathbf{Q}}_0 &= -\bar{J} \bar{k}_0 [\bar{\mathbf{F}}^{-1} \cdot \bar{\mathbf{F}}^{-t} \otimes \bar{\mathbf{F}}^{-t} + \bar{\mathbb{B}}] : \operatorname{Grad} \bar{\Theta}, \partial_{\operatorname{Grad} \bar{\Theta}} \bar{\mathbf{Q}}_0 = -\bar{J} \bar{k}_0 \bar{\mathbf{G}} \\
\bar{\mathbb{D}} &= \partial_{\bar{\mathbf{F}}} \bar{\mathbf{F}}^{-t} = -\bar{\mathbf{F}}^{-t} \underline{\otimes} \bar{\mathbf{F}}^{-1} + [\mathbf{i} - \bar{\mathbf{i}}] \underline{\otimes} \bar{\mathbf{F}}^{-1} \cdot \bar{\mathbf{F}}^{-t}, \bar{\mathbb{I}} = \partial_{\bar{\mathbf{F}}} \bar{\mathbf{F}} = \mathbf{i} \underline{\otimes} \bar{\mathbf{I}} \\
\bar{\mathbb{B}} &= \partial_{\bar{\mathbf{F}}} (\bar{\mathbf{F}}^{-1} \cdot \bar{\mathbf{F}}^{-t}) = {}^t [\bar{\mathbf{F}}^{-1} \cdot \bar{\mathbb{D}}] + \bar{\mathbf{F}}^{-1} \cdot \bar{\mathbb{D}}, \bar{\mathbf{G}} = \bar{\mathbf{F}}^{-1} \cdot \bar{\mathbf{F}}^{-t}
\end{aligned}$$

of the thermo-mechanically degrading, highly conductive interface on the overall response of the body, all thermo-mechanical properties of the bulk are fixed.

Interface conductivity. For this example, the prescribed displacement $d_{\max}^p = 1$ mm is applied in 20 equal steps, and the total time is 10 ms. The interface interactive radius $\bar{R}$ is taken to be 0.02 mm. The effective (undamaged) interface heat conduction coefficient $\bar{k}_0$ varies from 0 to 100 times that of the bulk's. The rest of the interface thermal properties are set to zero to better study the effects of interface conduction. Two cases are considered and compared for this example and the upcoming ones: interface with and without in-plane degradation.

Table 4. Material properties assumed in the numerical examples. Note that $\kappa = \lambda + 2/3 \mu$ and $\bar{\kappa} = \bar{\lambda} + \bar{\mu}$.

	Bulk		Interface	
Lamé constant	μ	80193.8 N/mm ²	$\bar{\mu}$	2×80193.8 N/mm
Lamé constant	λ	110743.5 N/mm ²	$\bar{\lambda}$	2×110743.5 N/mm
compression modulus	κ	164206.03 N/mm ²	$\bar{\kappa}$	2×190937.3 N/mm
initial temperature	Θ_0	298 K	$\bar{\Theta}_0$	298 K
specific heat capacity	c_F	3.588 N/[mm ² K]	$\bar{F}_0$	0.005
heat expansion coeff.	α	10^{-5} 1/K	$\bar{F}_f$	0.1
heat conduction coeff.	k	45 N/[sK]	$\bar{R}$	0.02 & 0.04 mm

Figure 4. Strip with curved interface, (a) geometry and (b) applied boundary conditions. Dimensions are in mm. The thickness is 0.05. **Figure 5.** The bulk temperature distribution of the strip at 100% of the final deformation for $\bar{\mu}/\mu = \bar{\lambda}/\lambda = 2$ mm and $\bar{k}_0/k = 0 - 100$ mm. Results (a)–(d) correspond to the intact interface, whereas results (e)–(h) correspond to the degrading interface.

Figure 5 shows the temperature distribution of the above cases at the end of the applied displacement. By increasing the heat conduction coefficient a more uniform temperature distribution along the intact interface is achieved (see Figure 5(a)–(d)) as opposed to the temperature distribution along the degraded interface (see Figure 5(e)–(h)). Moreover, since the stiffness of the interface is also affected by the damage evolution, the damaged interface is more deformed compared to its undamaged counterpart (compare Figure 5(h) with (d)). It is worthwhile to mention that the interface in this example is mechanically energetic and thermal. Thus the damage variable is solely dependent on the interface deformation gradient. Figure 6 depicts the evolution of

Figure 6. The evolution of the temperature (a)–(e) and x-component of the Cauchy heat flux vector (f)–(j) on the degrading interface for $\bar{\mu}/\mu = \bar{\lambda}/\lambda = 2$ mm and $\bar{k}_0/k = 100$ mm.

Figure 7. The bulk temperature distribution of the strip at 100% of the final deformation for $\bar{\mu}/\mu = \bar{\lambda}/\lambda = 2$ mm and $\bar{\alpha}/\alpha = 0-1.5$. Results (a)–(d) correspond to the intact interface, whereas results (e)–(h) correspond to the degrading interface.

the temperature and x-component of the Cauchy heat flux vector on the degrading interface. It is observed that at higher deformation levels, hence higher degradation, the conductivity along the interface is noticeably reduced (see Figure 6(a)–(d)), which in turn decreases the magnitude of the heat flux in the bulk and alleviates its discontinuity intensity (see Figure 6(e)–(h)). This observation can be explained by the temperature evolution equation (7) which for the current case simplifies to

$$[\![\mathbf{Q}]\!] \cdot \bar{\mathbf{N}} = -\bar{\text{Div}} \bar{\mathbf{Q}} \text{ with } \bar{\mathbf{Q}} = [1 - \bar{D}_{\parallel}] \bar{\mathbf{Q}}_0.$$

Gough–Joule effect. In this example, the prescribed displacement $d_{\max}^p = 1$ mm is applied in 20 equal steps, and the total time is 10 ms. The interface interactive radius $\bar{R}$ is taken to be 0.02 mm. The heat expansion coefficient is increased from zero to 1.5 times that of the bulk's. The remaining interface thermal parameters are set to zero.

Due to the Gough–Joule effect the intact interface cools under tensile load (see Figure 7(a)–(b)). However, by including the in-plane degradation, the interface cooling is not as pronounced as that of the intact interface, since now the cooling is proportional to $[1 - \bar{D}_{\parallel}] \bar{\alpha}$. This effect is shown in Figure 7(e)–(h). Figure 8 depicts the evolution of the temperature and the x-component of the Cauchy heat flux vector on the degrading interface. By increasing the loading the overall interface temperature decreases. Nonetheless the temperature reduction is

Figure 8. The evolution of the temperature (a)–(e) and the x-component of the Cauchy heat flux vector (f)–(j) on the degrading interface for $\bar{\mu}/\mu = \bar{\lambda}/\lambda = 2$ mm and $\bar{\alpha}/\alpha = 1.5$.

limited by the damage growth (see Figure 8(a)–(e)). At higher deformation levels, thus higher damage values, the magnitude of the jump of heat flux across the interface is reduced. This matter is depicted in Figure 8(f)–(j) and can be explained by the temperature evolution equation (7) which in this case simplifies to

$$\begin{aligned} [\mathbf{Q}] \cdot \bar{\mathbf{N}} &= \bar{\Theta} \partial_{\bar{\Theta}} ([1 - \bar{D}_{\parallel}] \bar{\mathbf{P}}_0) : \mathbf{D}_t \bar{\mathbf{F}} \\ &= [\bar{\Theta} [1 - \bar{D}_{\parallel}] \partial_{\bar{\Theta}} \bar{\mathbf{P}}_0 - \bar{\Theta} \partial_{\bar{\Theta}} \bar{D}_{\parallel} \bar{\mathbf{P}}_0] : \mathbf{D}_t \bar{\mathbf{F}} \\ &= -\bar{\Theta} [1 - \bar{D}_{\parallel}] 2\bar{\alpha} \bar{\kappa} [1 - \ln \bar{J}] / \bar{J}^{-1} \operatorname{div} \bar{\mathbf{V}} \\ &\quad - \bar{\Theta} \bar{\mathbf{P}}_0 : \mathbf{D}_t \bar{\mathbf{F}} \bar{\mathcal{D}}'_{\parallel} \int_{\mathcal{I}_0} \omega_{rs} \frac{[-2\bar{\alpha} \bar{\kappa} \bar{J}^{-1} \ln \bar{J}]_{\bar{\mathbf{x}}_s}}{\bar{E} \bar{F}_{\text{loc}}(\bar{\mathbf{x}}_s)} dA_s, \end{aligned}$$

where $\omega_{rs} = \omega(\bar{\mathbf{x}}_r, \bar{\mathbf{x}}_s)$. It is important to point out that at high enough deformation, $[1 - \bar{D}_{\parallel}] \approx 0$ and $\bar{\mathcal{D}}'_{\parallel} \approx 0$, as suggested by the damage function (16). Thus the jump of the heat flux across the interface is reduced as the damage evolves.

Interface heat capacity. For this example, the prescribed displacement $d_{\max}^p = 0.3$ mm is applied in 300 equal steps⁴, and the total time is 10 ms. The interface interactive radius $\bar{R}$ is taken to be 0.04 mm. The interface effective heat expansion coefficient $\bar{c}_{\bar{\mathbf{F}}}^0$ ranges from zero to 1 times that of the bulk. The other thermal parameter are set to zero.

The temperature distributions of the intact and degraded interface are shown in Figure 9. We need to mention that the heat capacity coefficient determines the thermal inertia of the material. Thus, the higher the interface heat capacity, the more resistant the interface is to a temperature change. On the intact interface (see Figure 9(a)–(b)), increasing $\bar{c}_{\bar{\mathbf{F}}}^0$ causes less temperature decrease on the interface. This, however is no longer the case on the degrading interface, where as the damage evolves, the nominal heat capacity coefficient $\bar{c}_{\bar{\mathbf{F}}} = [1 - \bar{D}_{\parallel}] \bar{c}_{\bar{\mathbf{F}}}^0$ is reduced. This is the reason why in Figure 9(e)–(h) a dramatic reduction of the interface temperature is observed. Note that in the regions where the interface is not yet fully damaged, higher temperatures are maintained (see Figure 9(h)). Figure 10 depicts the evolution of the temperature and x-component of the Cauchy heat flux vector on the degrading interface. It is observed that with the onset of damage the degrading interface begins to cool down by losing its thermal inertia (see Figure 10(a)–(e)). Note that increasing the deformation results in an increasing jump in the Cauchy heat flux across the interface in the regions on the interface that are not fully damaged. In the degraded regions of the interface the jump of heat flux across the interface vanishes (see Figure 10(f)–(j)). The reason for this observation is founded on the following relation:

$$[\mathbf{Q}] \cdot \bar{\mathbf{N}} = -\bar{c}_{\bar{\mathbf{F}}} \mathbf{D}_t \bar{\Theta} \text{ with } \bar{c}_{\bar{\mathbf{F}}} = [1 - \bar{D}_{\parallel}] \bar{c}_{\bar{\mathbf{F}}}^0,$$

which is the simplified interface temperature evolution equation for this example.

Figure 9. The bulk temperature distribution of the strip at 100% of the final deformation for $\bar{\mu}/\mu = \bar{\lambda}/\lambda = 2$ mm and $\bar{c}_F^0/c_F = 0 - 1$ mm. Results (a)–(d) correspond to the intact interface, whereas results (e)–(h) correspond to the degrading interface.

Figure 10. The evolution of the temperature (a)–(e) and x-component of the Cauchy heat flux vector (f)–(j) on the degrading interface for $\bar{\mu}/\mu = \bar{\lambda}/\lambda = 2$ mm and $\bar{c}_F^0/c_F = 1$ mm.

6. Summary and conclusion

A theoretical and computational framework for continua containing geometrically and thermally coherent (highly conductive) interfaces with tangential thermo-mechanical resistance on the interface was presented. The interface is endowed with its own thermo-mechanically energetic structure. The interface is thermally highly conductive, allowing for the jump in the normal heat flux across the interface but not in the temperature field. The corresponding mechanical and thermal weak forms of the balance equations were given. The balance equations were fully discretized using the finite element method in space. The effect of the in-plane degradation on the thermo-mechanical properties of the interface and overall response of the body were also taken into account by introducing a tangential reduction factor. For doing so, a non-local damage model is chosen, in which the equivalent distortion is non-localized using integral-type averaging. The equivalent distortion is a function of the interface thermo-mechanical effective Helmholtz energy, which consequently makes the damage variable depend on both the interface deformation and temperature.

A series of numerical examples served to elucidate the role of the thermo-mechanically energetic and thermally highly conductive interface, subject to in-plane degradation, on the overall response of the body. It was shown that the degraded interface undergoes more deformation. In addition, due to its reduced heat conduction coefficient, higher temperature gradients along the interface were observed. It was also illustrated that interface

cooling due to the Gough–Joule effect is limited by the reduced heat expansion coefficient. Moreover, the interface material degradation strongly affects the thermal inertia of the interface as represented by the heat capacity coefficient. The damaging interface with the reduced heat capacity coefficient was shown to be less capable of being resistant to temperature changes. Finally in all the examples, asymptotically quadratic convergence associated with the Newton–Raphson scheme was achieved.

One consequent extension to this work is to study the role of the tangential degradation of the interface material on the thermo-mechanical response of a general imperfect interface. It is also of great interest to include geometrical non-coherency by allowing a displacement jump across the interface, which requires the use of a cohesive zone model. These extensions shall be discussed in later contributions.

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Notes

1. The integral term in equation (8) is introduced in analogy with that of Simo and Hughes [52, Section. 1.3.3] and is the energy storage in the material due to the accumulation of microscopic defects.
2. The geodesics are the general form of straight lines when applied to curved, three-dimensional interfaces. The minimal geodesics in differential geometry are the shortest distance paths between two points on an interface.
3. In what follows, for the sake of brevity, homogeneous Neumann boundary conditions are assumed and hence, some integrals vanish. The integrals are standard and require no additional care due to the thermo-mechanically energetic highly conductive interface subject to in-plane degradation, which is the focus of this contribution.
4. In this example the maximum stretch is reduced and the number of steps is increased resulting in the smaller prescribed displacement per increment. These measures are taken to deal with the numerical issues arising from the reduced interface heat capacity coefficient which causes a high temperature gradient along the interface. The existence of a high temperature gradient in turn gives rise to spurious stress oscillations, deterring the convergence of the numerical simulation.

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