Modular Deformations of Isolated Singularities

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Introduction

When studying a category of objects (in algebraic or analytic geometry), an important aim is to achieve a better understanding of the objects under consideration by trying to classify them. In singularity theory, a tremendous effort has been undertaken by V. I. Arnol'd and many others which lead to lists of normal forms for some classes of singularities – for a survey, see [AGZV85] or [AGLV98], for instance. In general, when the classes of objects become more complicated, one cannot hope to find complete classification lists anymore. A different approach is to fix certain invariants and try to construct a moduli space for singularities of the given type, so in particular one tries to find a universal family containing as fibres over the base space all objects under investigation. For some recent works in that direction see [GHP97], [FK00], for example.

Nevertheless, one major obstacle that one encounters when trying to construct moduli spaces for geometric objects is that, in general, such universal families do not exist: The best one can get is a *semi*-universal deformation, which usually is not universal – in the language of Schlessinger's classical paper [Sch68]: the corresponding deformation functors admit a hull but are not representable. One possible approach is thus to single out some sort of maximal universal locus in the base space of a given semi-universal deformation. This idea had first been introduced by V. P. Palamodov in [Pal78] in the framework of deformations of compact complex spaces, following him we call such a subspace the *modular stratum* of the object we started with. The similar notion of *prorepresenting substratum* was investigated in the context of formal deformations of affine schemes by Laudal and Pfister in [LP88].

The existence of maximal modular subspaces has been established for a number of deformation problems in analytic geometry. A very general existence statement can be found in [KS90], [Pal02] contains a recent, more geometric treatment. Nevertheless, very few concrete examples of modular strata are known: A. G. Aleksandrov has shown in [Ale85] that isolated quasihomogeneous complete intersections in positive dimension have reduced and smooth modular stratum, and there are several examples of singularities where the modular stratum is non-reduced and Artinian. The aim of the present thesis is to describe an algorithmic method for determining modular strata by means of a computer, and to use its implementation for the computation of a series of non-trivial examples. This is built on recent work of B. Martin: By showing that the modular stratum of an isolated singularity X_0 can be computed as the flattening stratum of a suitable module over the local ring of the base space of a semi-universal deformation of X_0, we can use his algorithm to compute local flattenings presented in [Mar02a].

In particular, first examples had shown that the singular locus of the original singularity may split along the modular stratum, so one is led to the question of whether such a phenomenon still can happen when considering deformations with section instead. Therefore we include in our treatment not only the 'usual' deformation functor of a singularity but also the

functor of deformations with section.

In chapter one, we begin by summarizing the technical background. In the study of deformation theory, the most important algebraic tool is provided by cotangent cohomology, so we start by reviewing the main constructions and properties of the cotangent cohomology of analytic algebras and their morphisms, as it had been developed by Palamodov [Pal76] and Flenner [Fle78] as an analogue of the so-called André-Quillen homology of commutative rings (cf. [Qui70], [And74]). Having introduced them, we recall the basic notions of deformation theory, in particular treating the functors of deformations with section and singular section that go back to Buchweitz' thesis [Buc81] in some more detail.

The second chapter then introduces the main concept of this thesis: Modular subspaces. After summing up the basic definitions and properties, we give a proof of the existence of maximal modular subspaces in the base space of a semi-universal deformation of an isolated singularity, mainly using the constructions of [Pal78] and [KS90]. In particular we show (Theorem 2.2.4): If $X \to S$ is a semi-universal deformation of the singularity X_0, then $M \subseteq S$ is modular if and only if one of the following two equivalent criteria is fulfilled:

(1) The relative Kodaira-Spencer map $T^0(S, \mathcal{O}_M) \to T^1(X_{|M}/M)$ is injective.

(2) The evaluation map $T^0(X_{|M}/M) \to T^0(X_0)$ of relative vector fields of the restricted deformation to vector fields of the special fibre is surjective,

and an analogous criterion is derived for deformations with section (Theorem 2.2.4). As a first application we show that Aleksandrov's smoothness result for the modular stratum of quasihomogeneous complete intersections can be carried over to the functor of deformations with section, in particular we obtain that the modular strata of both functors coincide for such singularities.

The latter of the above characterizations of modular subspaces admits an interpretation as flatness of the $\mathcal{O}_M$-module $\tilde{T}^1(X_{|M}/M) := (\mathcal{O}_X^k/J(F)) \otimes_{\mathcal{O}_S} \mathcal{O}_M$, where $F = F_1, \ldots, F_k$ are equations defining $X \subseteq S \times \mathbb{C}^n$ and $J(F)$ stands for the Jacobian matrix with respect to coordinates on $\mathbb{C}^n$. In particular we show that for unobstructed singularities, i. e. $T^2(X_0) = 0$, this is actually equivalent to $\mathcal{O}_M$-flatness of the module of infinitesimal relative deformations $T^1(X_{|M}/M)$. This agrees with the fact that the modular stratum coincides with the τ-constant stratum in the base space of a semi-universal deformation, i. e., for suitably chosen representatives of the germs, $M \subseteq S$ coincides (as set) with those points $s \in S$ for which the deformation is still a semi-universal deformation of the fibre over s.

Having all these theoretical criteria at our disposal, we can then describe an algorithm to compute the modular stratum in chapter three. Its starting point is the description of the tangent space to the modular stratum M_{X_0} of the singularity X_0 as

$$T(M_{X_0}) = \{t \in T(S) : [\delta, \theta_\xi(t)] = 0 \text{ for all } \delta \in T^0(X_0)\},$$

where $\xi : X \to S$ is a semi-universal deformation of X_0, $\theta_\xi : T(S) \to T^1(X_0)$ denotes the Kodaira-Spencer map and $[-,-] : T^0(X_0) \times T^1(X_0) \to T^1(X_0)$ is the Lie bracket in cotangent cohomology. Having computed the tangent space, i. e. a first-order approximation of M_{X_0}, we try to find the maximal extension in the next order that preserves flatness of $\tilde{T}^1(X/S)$. This is done by methods similarly to those described in [Mar02a]. This approach makes it possible to compute d-jets of equations defining the ideal of the modular stratum inside $\mathcal{O}_S$.

To a certain extent this approach can be seen as an analogue to the algorithmic construction of a semi-universal deformation, cf. [Mar99] – in this case one has to kill obstructions in $T^2(X_0)$, whereas in our situation the obstruction space is given by $T^1(X_0)$.

The resulting algorithm has been implemented (for both deformation functors) in the computer algebra system SINGULAR [GPS02] as library `modular.lib` [HM04]. This algorithm and its implementation generalize first versions described in [Mar02a], [Nik03].

After describing the implementation by means of some first examples, we exhibit in the closing chapter four a series of phenomena that can occur in the modular stratum and have been found using SINGULAR: First of all, the answer to the question raised above is negative: The singular locus may split along a modular family even when restricting to deformations with section. – In fact, the same singularity where this happens for the ordinary deformation functor provides an example for the other functor, too. Besides further concrete examples, we also present a complete description of the modular strata of all unimodal hypersurface singularities, in particular showing that the singularities of the $T_{p,q,r}$-series possess an Artinian modular stratum in general, with exceptions only those singularities that are adjacent to the so-called parabolic unimodal hypersurfaces.

Further examples investigate quasihomogeneous and semi-quasihomogeneous singularities: Taking Aleksandrov's above cited result as a starting point (the fact that quasihomogeneous complete intersections of positive dimension have reduced and smooth modular stratum), we show that the converse of this assertion is wrong: There are non-quasihomogeneous hypersurfaces whose modular stratum is smooth as well. In addition, we present examples which demonstrate that it also becomes wrong as soon as one of its assumptions (quasihomogeneity, positive dimension, complete intersection) is no longer fulfilled. We close this discussion by formulating several conjectures concerning the modular stratum of semi-quasihomogeneous hypersurface singularities, which are justified by their verification in certain special cases.

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Zusammenfassung

Bei der Untersuchung von Objekten in der (algebraischen bzw. analytischen) Geometrie gehört es zu den wichtigsten Zielen, durch die Klassifikation der betrachteten Objekte ein besseres Verständnis ihrer Struktur zu erlangen. In der Singularitätentheorie wurden von V. I. Arnol'd und vielen anderen eine Reihe solcher Klassifikationsresultate erzielt – eine Übersicht findet sich zum Beispiel in den Monographien [AGZV85] und [AGLV98]. Der Versuch, durch Angabe von Normalformen vollständige Typisierungen zu erhalten, ist jedoch nicht mehr erfolgversprechend, wenn die betrachteten Objekte komplizierter werden. In dieser Situation ist es sinnvoll, stattdessen gewisse Invarianten zu fixieren und zu versuchen, Modulräume für die Singularitäten des gegebenen Typs zu konstruieren. Insbesondere beinhaltet dieser Zugang die Bestimmung einer universellen Familie, deren Fasern über dem Basisraum gerade Repräsentanten der Isomorphieklassen der betrachteten Objekte sind. Als Beispiele für jüngere Arbeiten in dieser Richtung seien [GHP97], [FK00] genannt.

Im allgemeinen existieren solche universellen Familien von Singularitäten jedoch nicht – in der Sprache der Deformationstheorie ausgedrückt: Zu jeder isolierten Singularität X_0 existiert nach einem grundlegenden Resultat von H. Grauert [Gra72] zwar eine semi-universelle Deformation, es ist aber im allgemeinen nicht möglich, eine universelle Deformation zu erhalten. Ein möglicher Ansatz, der sich erstmals in V. P. Palamodovs Arbeit [Pal78] im Zusammenhang mit der Untersuchung von Deformationen kompakter komplexer Räume findet, ist nun, den Basisraum einer semi-universellen Deformation $X \to S$ auf einen maximalen Unterkeim $M \subseteq S$ einzuschränken, so daß die universelle Eigenschaft zumindest für alle aus M induzierten Familien erfüllt ist. $M \subseteq S$ wird dann das *Modularstratum* der ursprünglichen Singularität X_0 genannt.

Die Existenz maximaler modularer Unterkeime ist für eine Reihe von Deformationsproblemen in der analytischen Geometrie gesichert, vgl. zum Beispiel [KS90] oder [Pal02]. Andererseits ist, von einer Charakterisierung im Falle quasihomogener vollständiger Durchschnitte durch A. G. Aleksandrov [Ale85] abgesehen, bislang nur das Modularstratum von wenigen Singularitäten bekannt. Ziel der vorliegenden Arbeit ist es daher, einen Algorithmus zu seiner Berechnung zu beschreiben und eine darauf beruhende Implementierung in dem Computeralgebra-System Singular [GPS02] zur Berechnung einer Reihe nicht-trivialer Beispiele zu nutzen.

In [Mar02a] wurde ein Beispiel vorgestellt, in dem der singuläre Ort der ursprünglichen Singularität entlang einer modularen Familie in mehrere singuläre Punkte zerfällt. Dies führt auf die Frage, ob ein solches Phänomen auch auftreten kann, wenn anstelle des üblichen Deformationsfunktors der Funktor von Deformationen mit Schnitt betrachtet wird. Aus diesem Grund werden beide Funktoren in dieser Arbeit weitgehend parallel behandelt.

Im ersten Kapitel werden die benötigten technischen Hilfsmittel bereitgestellt: Das alge-

braische Hilfsmittel zur Untersuchung von Deformationen analytischer Mengenkeime liefern der Kotangentenkomplex und die daraus berechnete Kotangentialkohomologie lokaler analytischer Algebren und ihrer Morphismen. Die notwendigen und später verwendeten diesbezüglichen Begriffe und Resultate werden hier zusammengestellt. Weiterhin werden die Grundlagen aus der Deformationstheorie zusammengetragen – einerseits, damit sie in den Folgekapiteln zur Verfügung stehen, andererseits werden insbesondere Deformationen mit (singulärem) Schnitt auf Grundlage der Arbeit von R. O. Buchweitz [Buc81] ausführlicher behandelt.

Im zweiten Kapitel wird zunächst der Begriff eines modularen Unterkeimes M in der Basis S einer Deformation $\xi : X \to S$ der Singularität X_0 eingeführt. Nach der Formulierung grundlegender Eigenschaften werden eine Reihe von äquivalenten Bedingungen bewiesen, unter denen $M \subseteq S$ modular ist. Insbesondere wird gezeigt, daß dies für semi-universelles ξ genau dann der Fall ist, wenn

(1) die relative Kodaira-Spencer-Abbildung $T^0(X/S, \mathcal{O}_M) \to T^1(X/S)_{|M}$ injektiv oder

(2) die Auswertungsabbildung $T^0(X/S)_{|M} \to T^0(X_0)$ surjektiv ist.

Weiter wird daraus folgendes Kriterium abgeleitet, das die Grundlage für den im Folgekapitel vorgestellten Algorithmus zur Berechnung des Modularstratums darstellt: Wird der Totalraum X einer semi-universellen Deformation $X \to S$ von X_0 durch $F_1, \ldots, F_k \in \mathcal{O}_S \otimes \mathbb{C}\{x_1, \ldots, x_n\}$ definiert, dann ist $M \subseteq S$ genau dann modular, wenn

$$\tilde{T}^1(X/S)_{|M} := (\mathcal{O}_X^k / J(F)) \otimes_{\mathcal{O}_S} \mathcal{O}_M$$

ein flacher $\mathcal{O}_M$-Modul ist, wobei $J(F)$ die relative Jacobi-Matrix bezüglich $x_1, \ldots, x_n$ bezeichnet.

Ferner wird gezeigt, daß für unobstruiertes X_0 diese Bedingungen dazu äquivalent sind, daß der relative Tjurina-Modul $T^1(X/S)_{|M}$ ein flacher und damit freier $\mathcal{O}_M$-Modul (vom Rang $\tau := \dim_{\mathbb{C}} T^1(X_0)$) ist. Dies stimmt mit dem Resultat überein, daß das Modularstratum solcher Singularitäten dem Stratum $\tau = const$ entspricht. Analoge Charakterisierungen werden für Deformationen mit Schnitt hergeleitet. Auf dieser Grundlage kann im dritten Kapitel ein Algorithmus beschrieben werden, der die Berechnung des Modularstratums M_{X_0} von X_0 (zumindest bis zu einer gewissen Ordnung) ermöglicht: Ausgangspunkt ist die Charakterisierung seines Tangentialraumes als

$$T(M_{X_0}) = \{t \in T(S) : [\delta, \theta_\xi(t)] = 0 \text{ für alle } \delta \in T^0(X_0)\},$$

wobei $\theta_\xi : T(S) \to T^1(X_0)$ die Kodaira-Spencer-Abbildung der semi-universellen Deformation $\xi : X \to S$ bezeichnet, und $[-,-] : T^0(X_0) \times T^1(X_0) \to T^1(X_0)$ die Lie-Klammer der Kotangentialkohomologie. Der erste Schritt besteht also in der Berechnung dieser Daten, womit man relativ einfach den Tangentialraum des Modularstratums erhält. Hat man es nun bis zu einer gewissen Ordnung bestimmt, dann erhält man induktiv die Approximation nächster Ordnung durch Berechnung der maximalen Erweiterung, so daß die Flachheit von $\tilde{T}^1(X/S)$ erhalten bleibt. Die algorithmische Umsetzung dieses Ansatzes läßt sich nun weitgehend mit den in [Mar02a] beschriebenen Methoden erreichen. Schließlich wird die Implementierung dieses Algorithmus' als SINGULAR-Bibliothek `modular.lib` [HM04] beschrieben.

Im letzten Kapitel werden nun unter Verwendung dieser Bibliothek eine Reihe von nichttrivialen Beispielen für Modularstraten berechnet. Zunächst wird die oben aufgeworfene Frage

(negativ) beantwortet: Ein Beispiel zeigt, daß ein Zerfallen des singulären Ortes auch möglich ist, wenn Deformationen mit Schnitt betrachtet werden. Des weiteren wird eine vollständige Beschreibung des Modularstratums aller null- und unimodalen Hyperflächensingularitäten gegeben. Dabei wird insbesondere bewiesen, daß dieses bei den Singularitäten der $T_{p,q,r}$-Serie bis auf Ausnahmen stets nulldimensional ist.

Weitere Beispiele untersuchen quasihomogene und semi-quasihomogene Singularitäten, hier wird Aleksandrovs oben zitierte Beschreibung des Modularstratums quasihomogener vollständiger Durchschnitte (positiver Dimension) als Ausgangspunkt genommen: Für solche Singularitäten ergibt sich stets ein glatter, reduzierter Keim. Es wird gezeigt, daß die Umkehrung dieser Aussage nicht gilt (es gibt zum Beispiel nicht-quasihomogene Hyperflächen mit glattem Modularstratum), und daß sie auch falsch wird, wenn eine der Voraussetzungen (Quasihomogenität, positive Dimension, vollständiger Durchschnitt) nicht mehr erfüllt ist. Die Arbeit schließt mit der Formulierung mehrerer Vermutungen, die sich aus der Berechnung einer Reihe von Beispielen semi-quasihomogener Hyperflächensingularitäten ergeben haben.

Contents

Chapter 1

Cotangent complex and deformations

The aim of this introductory chapter is to lay out the groundwork for the subsequent chapter two, where we will need the basic notions and results of the deformation theory of complex analytic germs and, in particular, their cotangent cohomology as the main technical tool in this context. The results presented below are mostly standard, but appear somewhat scattered through the literature. Therefore we give an account of these facts which we will refer to in the following chapters two and three.

In section 1.3, where we study the functors of deformations with section and with singular section of a singularity, we will recall the basic constructions of Buchweitz' thesis [Buc81], where the correct cotangent cohomology modules controlling deformations with section are determined and semi-universal deformations for both functors are constructed. Our contribution is to give a somewhat more detailed treatment of the corresponding cohomology modules, in particular in terms of defining equations, which will become important later on when we want to do explicit computations.

1.1 Cotangent cohomology

In this section we recall the construction of the cotangent (co-)homology modules of an analytic algebra and summarize of their properties. In the category of commutative algebras, this goes back to the work by Lichtenbaum and Schlessinger who defined the cotangent complex of a morphism in degrees 0, 1 and 2 in [LS67], this has then been extended by Quillen in [Qui70]. A different approach by André [And74], based on so-called simplicial resolvents, yields the same result, and the corresponding homology theory is called *André-Quillen homology*. A gentle introduction to this homology theory and its application to the deformation theory of affine singularities can be found in [Man01], for example.

The complex-analytic version of this theory is an analogue of André-Quillen homology – not just a special case since one has to take into account the topology of analytic algebras and the distinction between the analytic tensor product (cf. [GR71]) and the usual one, see e. g. the first two sections of [Pal82] for a detailed discussion. The cotangent complex and cotangent cohomology of analytic algebras and complex spaces are introduced by Palamodov in [Pal76], based on earlier unpublished work of Tjurina. It is studied further by Flenner and Bingener in [Fle78] resp. [Bin80]. It is this theory that we are now going to review.

Kramm and Schuster have given a different construction in [KS77], using a variant of André's simplicial methods. A proof of the equivalence of both approaches can be found in [Pal82].

One third construction is worth being mentioned: If B is a flat (analytic) A-algebra, then cotangent cohomology is (up to a degree shift by one) a direct summand of Hochschild cohomology, see [Lod98] or [Wei94] for a proof in the affine and [Pal82] in the analytic case.

To this end, we start by defining the cotangent complex of a morphism $A \to B$ of analytic algebras – or, equivalently, a morphism $X \to S$ of germs of complex spaces*. We do not include proofs for most of the following statements but only give references where they can be found.

1.1.1 Differential graded algebras

Definition 1.1.1.

(1) A *graded anti-commutative analytic algebra* is an associative graded ring $R = \bigoplus_{i \leq 0} R^i$ with unit such that

i) $xy = (-1)^{ij} yx \in R^{i+j}$ for all $x \in R^i$ and $y \in R^j$,

ii) R^0 is an analytic algebra and

iii) for all i, R^i is a finite R^0-module.

A *homomorphism* $\varphi : R \to S$ between such algebras is defined as a homomorphism of rings such that $\varphi(R^i) \subset S^i$ for all i and $\varphi_{|R^0} : R^0 \to S^0$ is a homomorphism of analytic algebras.

(2) If R is a graded anti-commutative analytic algebra, a *graded R-module* is an R-bimodule M with a grading $M = \bigoplus_{i \in \mathbb{Z}} M^i$, such that

$$xm = (-1)^{ij} mx \in M^{i+j}$$

for all $x \in R^i$ and $m \in M^j$.

A *homomorphism* $\varphi : M \to N$ *of graded R-modules* of degree p is a homomorphism of right-modules such that $\varphi(M^n) \subseteq N^{n+p}$ for all $n \in \mathbb{Z}$.

Remark 1.1.2. Let us mention some standard constructions:

(1) If M and N are graded R-modules, then $M \otimes_R N$ inherits a graded bimodule structure in a natural way.

(2) If $\mathrm{Hom}_R(M, N)^p$ denotes the set of homomorphisms $M \to N$ of degree p, then the direct sum $\mathrm{Hom}_R(M, N) := \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_R(M, N)^p$ again forms a graded R-module.

(3) For any homomorphism $R \to S$ of graded anti-commutative analytic algebras, there exists a universal finite graded S-module $\Omega_{S/R}$ of differentials satisfying the usual universal property, i. e.

$$\mathrm{Der}_R(S, M) \simeq \mathrm{Hom}_S(\Omega_{S/R}, M)$$

*Since we are dealing almost exclusively with germs, we will usually omit the distinguished point of a germ and simply write $X := (X, 0)$ etc.

for any graded S-module M which is separated as an S^0-module, and $\Omega^i_{S/R}$ is a finite S^0-module for all i. Here a *derivation* $\delta \in \mathrm{Der}_R(S, M)$ is an R-linear map $S \to M$ such that

$$\delta(xy) = \delta(x) \cdot y + (-1)^{ij}\delta(y) \cdot x$$

for homogeneous $x \in S^i$ and $y \in S^j$ (cf. [Fle78, § 1 A.], where also further examples and properties of modules over such algebras are studied).

Definition 1.1.3.

(1) A *differential graded algebra (DG-algebra)* is a graded anti-commutative analytic algebra R, together with an R^0-derivation $s : R \to R$ of degree 1 satisfying $s^2 = 0$.

A *homomorphism of DG-algebras* is a homomorphism of graded anti-commutative analytic algebras which is compatible with the respective differentials.

(2) A *differential graded module (DG-module)* over a DG-algebra R with differential s is a graded R-module M together with a differential $M \to M$ of degree 1 (which, by abuse of notation, is also denoted s) satisfying

$$s(xm) = s(x)m + (-1)^{ij}s(m)x$$

for homogeneous elements $x \in R^i$ and $m \in M^j$.

In particular, if M and N are DG-modules, then $M \otimes_R N$ and $\mathrm{Hom}_R(M, N)$ carry a DG-module structure as well.

For further use we finally state the following definition:

Definition 1.1.4. A *differential graded Lie algebra (DG-Lie algebra)* over a DG-algebra R is a DG-module L together with a homogeneous bilinear operation $[-,-] : L \times L \to L$, called the *graded Lie bracket*, such that for homogeneous elements $x \in L^i$, $y \in L^j$ and $z \in L^k$ the following identities are satisfied:

i) $[x, y] + (-1)^{ij}[y, x] = 0$,

ii) $[x, [y, z]] + (-1)^{i(j+k)}[y, [z, x]] + (-1)^{k(i+j)}[z, [x, y]] = 0$ (Jacobi identity),

iii) $s([x, y]) = [s(x), y] + (-1)^i[x, s(y)]$.

1.1.2 The cotangent complex

We need the above notions in order to be able to define what a resolvent of a morphism of analytic algebras should be:

Definition 1.1.5. Let $A \to B$ be a morphism of analytic algebras. A *resolvent* for B over A is a DG-algebra R together with a surjective homomorphism $p : R \twoheadrightarrow B$ over A, such that

i) R is a free A-algebra and

ii) p is a quasi-isomorphism of complexes, i. e. $H^n(R) = 0$ for $n < 0$ and $H^0(R) \simeq B$, and this isomorphism is induced by p.

Theorem 1.1.6. *Any morphism of analytic algebras has a resolvent.*

Proof. [Pal76, Proposition 1.1] or [Fle78, 1.4]. ∎

Definition 1.1.7. Let $A \to B$ be a morphism of analytic algebras and R a resolvent for B over A.

(1) The complex
$$\mathbb{L}^{\bullet}_{B/A} := \Omega_{R/A} \otimes_R B$$
(in the derived category of the category of B-modules) is called the *cotangent complex* of B over A.

(2) For any B-module M we define the *cotangent cohomology modules* of B over A with values in M as
$$T^i(B/A, M) := \operatorname{Ext}^i_B(\mathbb{L}^{\bullet}_{B/A}, M), \qquad i \geq 0,$$
which defines the covariant *cotangent cohomology functors*.

This definition is justified by the following

Proposition 1.1.8. *Up to homotopy the cotangent complex of an analytic algebra does not depend on the choice of the resolvent.*

In particular, the cotangent cohomology modules are well-defined.

Proof. [Pal82, Proposition 3.4]. ∎

Notation. In order to simplify statements we adopt to the conventions
$$T^i(B/A) := T^i(B/A, B), \qquad T^i(B, M) := T^i(B/\mathbb{C}, M).$$
In addition, if $X \to S$ is a morphism of germs of complex spaces and M is an $\mathcal{O}_X$-module, we also write
$$T^i(X/S, M) := T^i(\mathcal{O}_X/\mathcal{O}_S, M).$$

Remark 1.1.9. Similarly, one can define tangent homology functors by setting
$$T_i(B/A, M) := \operatorname{Tor}^B_{-i}(\mathbb{L}^{\bullet}_{B/A}, M)$$
for an analytic A-algebra B and any B-module M, and most of the statements in cohomology below have their homological counterpart. But since we will not use these functors we do not discuss them further.

Proposition 1.1.10. *The functors T^i have the following properties:*

(1) $T^0(B/A, M) \simeq \operatorname{Der}_A(B, M)$.

(2) If $A \to B$ turns B into a regular A-algebra, then $T^i(B/A, M) = 0$ for $i > 0$.

(3) If $B \simeq A\{x\}/(f_1, \ldots, f_k)$, where $f_1, \ldots, f_k$ form a regular sequence in $A\{x\}$, then $T^i(B/A, M) = 0$ for $i > 1$.

(4) If $A \twoheadrightarrow B$ is a surjection with kernel I then $T^1(B/A, M) \simeq \operatorname{Hom}_B(I/I^2, M)$.

(5) If $T^1(B/A, \mathbb{C}) = 0$, then B is a regular A-algebra.

(6) If $T^2(B/A, \mathbb{C}) = 0$, then $B \simeq A\{x\}/(f_1, \ldots, f_k)$ with a regular sequence $f_1, \ldots, f_k \in A\{x\}$.

Proof. [Fle78, 1.15]. ∎

Remark 1.1.11. Note that (5) and (6) do not give the converse statements to (2) and (3), since coefficients are taken in different modules. In fact, $T^1(B/A) = 0$ does not imply that B is a regular A-algebra. For example, if $A = \mathbb{C}$ and $B = \mathbb{C}\{x\}/(f_1, \ldots, f_k)$, then B may be a non-regular analytic algebra but still $T^1(B) = 0$. In this case, the germ defined by $f_1, \ldots, f_k$ is called a *rigid* singularity – see e. g. [Sch73] for a discussion.

Similarly B may be a non-complete intersection but still $T^2(B) = 0$. This is, for instance, the case for Cohen-Macaulay germs of codimension 2 (see [Sch77] or [FK00], for example).

The functorial properties of cotangent cohomology are expressed in the following exact sequence which is immediate from the definition:

Proposition 1.1.12. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of B-modules. Then there is a long exact sequence in cohomology*

$$0 \to T^0(B/A, M') \to T^0(B/A, M) \to T^0(B/A, M'') \to T^1(B/A, M') \to \ldots$$

Now suppose we are additionally given a morphism $B \to C$, then we obtain an exact sequence

$$0 \to \mathbb{L}^\bullet_{B/A} \otimes_B C \to \mathbb{L}^\bullet_{C/A} \to \mathbb{L}^\bullet_{C/B} \to 0,$$

from which one derives:

Proposition 1.1.13. *If $A \to B \to C$ are homomorphisms of analytic algebras and M is a C-module, then there is a long exact sequence in cohomology*

$$0 \to T^0(C/B, M) \to T^0(C/A, M) \to T^0(B/A, M) \to T^1(C/B, M) \to \ldots$$

In particular, its beginning coincides with the canonical sequence

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M).$$

Proof. [Fle78, 1.14]. ∎

In particular, combining Proposition 1.1.10 and Proposition 1.1.13 we end up with the following sequence connecting T^0 and T^1:

Proposition 1.1.14. *Let $B \simeq A\{x\}/I$ be a quotient of a regular analytic A-algebra $A\{x\}$, and let M be a B-module. Then there is an exact sequence*

$$0 \to \mathrm{Der}_A(B, M) \to \mathrm{Der}_A(A\{x\}, M) \to \mathrm{Hom}_B(I/I^2, M) \to T^1(B/A, M) \to 0.$$

Proof. Applying Proposition 1.1.13 to the homomorphisms $A \hookrightarrow A\{x\} \twoheadrightarrow B$ we get the exact sequence

$$0 \to T^0(B/A, M) \to T^0(A\{x\}/A, M) \to T^1(B/A\{x\}, M) \to T^1(B/A, M) \to 0,$$

using property (2) in Proposition 1.1.10 and the fact that $T^0(B/A\{x\}, M) \simeq \mathrm{Der}_{A\{x\}}(B, M) = 0$. Now the statement follows from the identities (1) and (4) in Proposition 1.1.10. ∎

Remark 1.1.15.

- In fact, the sequence above results from applying $\mathrm{Hom}_B(-, M)$ to the so-called *conormal sequence*
$$I/I^2 \to \Omega_{A\{x\}/A} \otimes_{A\{x\}} B \to \Omega_{B/A} \to 0,$$
cf. [Eis95, ch. 16], for instance.

- The above description of the first cotangent cohomology module as cokernel of the map
$$\begin{aligned} \mathrm{Der}_A(A\{x\}, M) &\to \mathrm{Hom}_B(I/I^2, M) \\ \delta &\mapsto (f \bmod I^2 \mapsto \delta(f)) \end{aligned}$$
shows how to calculate it explicitly; in [Mar99] it is explained how the computation of $T^1(\mathbb{C}\{x\}/(f_1, \ldots, f_k))$ is performed in SINGULAR (procedure `T_1` in the standard library `sing.lib` [GM02]). In particular, if $f_1, \ldots, f_k$ form a regular sequence in $\mathbb{C}\{x_1, \ldots, x_n\}$, then
$$T^1(\mathbb{C}\{x\}/(f)) \simeq \mathbb{C}\{x\}^k/(f_i \cdot e_j)_{1 \le i,j \le k} + \left(\begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_k}{\partial x_1} \end{pmatrix}, \ldots, \begin{pmatrix} \frac{\partial f_1}{\partial x_n} \\ \vdots \\ \frac{\partial f_k}{\partial x_n} \end{pmatrix} \right).$$

- A similar explicit description can be given for the second cotangent cohomology module, cf. [dJP00, section 10.3]: Let $B = A\{x\}/I$ and $I = (f_1, \ldots, f_k)$. Denote by R the syzygy module of $f_1, \ldots, f_k$, i. e. the kernel of $A\{x\}^k \to A\{x\}, e_i \mapsto f_i$, and by R_0 the submodule of R generated by the Koszul relations $f_i e_j - f_j e_i$. Then
$$T^2(B/A, M) \simeq \mathrm{Hom}_{A\{x\}}(R/R_0, M)/\mathrm{Hom}_{A\{x\}}(A\{x\}^k, M),$$
and this can be computed in SINGULAR using the command `T_2`.

Finally, the following proposition describes the behavior of the T^i-functors under base change:

Proposition 1.1.16. *Let be given homomorphisms $A \to B$ and $A \to A'$ of analytic algebras, and set $B' := B \otimes_A A'$, i. e. we have a cocartesian diagram*
$$\begin{array}{ccc} B & \longrightarrow & B' = B \otimes_A A' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A'. \end{array}$$
If $A \to A'$ or $A \to B$ is flat, then there is a natural isomorphism
$$T^i(B/A, M) \simeq T^i(B'/A', M)$$
for any B'-module M and $i \ge 0$.

Proof. This follows from the fact that under these assumptions we have $\mathbb{L}^\bullet_{B'/A'} \simeq \mathbb{L}^\bullet_{B/A} \otimes_B B'$ [Fle78, 1.16]. ∎

Remark 1.1.17. If, in the situation above, both $A \to A'$ and $A \to B$ are flat, then there is also the identity $T^i(B/A, M) \otimes_A A' \simeq T^i(B'/A', M \otimes_A A')$ for any B-module M. If only $A \to A'$ is flat, still there is a map between these two B'-modules, which, however, in general needs not be an isomorphism.

We finish this section by noting how one can introduce a graded Lie algebra structure on $T^\bullet(B/A) := \bigoplus_{i \geq 0} T^i(B/A)$:

Proposition 1.1.18. *Let R be a resolvent of B over A. Then $\mathrm{Der}_A(R)$ becomes a DG-Lie algebra by setting*

$$[\delta_1, \delta_2] := \delta_1\delta_2 - (-1)^{(i+j)}\delta_2\delta_1$$

for homogeneous elements δ_1 and δ_2 of degrees i, j, respectively, and the differential in $\mathrm{Der}_A(R)$ is defined as $\delta \mapsto [s, \delta]$, where s stands for the differential in R. This bracket lifts to the cohomology $H^\bullet(\mathrm{Der}_A(R))$, which is isomorphic to $T^\bullet(B/A)$, hence inducing on $T^\bullet(B/A)$ the structure of a DG-Lie algebra.

Proof. cf. [Pal76, Proposition 1.10] or [Fle78, 1.21, 1.22]. ∎

Remark 1.1.19. If $B := A\{x\}/(f_1, \ldots, f_k)$ it is possible to describe the Lie bracket of $T^\bullet(B/A)$ in more concrete terms:

(1) $[-,-] : T^0(B/A) \times T^0(B/A) \to T^0(B/A)$ is just the ordinary commutator bracket of derivations in $\mathrm{Der}_A(B)$.

(2) Let $\tilde{\delta} \in \mathrm{Der}_A(A\{x\})$ induce an element $\delta \in T^0(B/A)$, so $\tilde{\delta}(f_i) = \sum_j h_{ij} f_j$ for $i = 1 \ldots k$. In addition, let $(g_1, \ldots, g_k) \in A\{x\}^k$ represent a class g in $T^1(B/A)$ (i. e. the class of the morphism $(f)/(f)^2 \to B$ defined by $\bar{f}_i \mapsto \bar{g}_i$). Then $[\delta, g] \in T^1(B/A)$ is the class in $T^1(B/A)$ of the morphism given by

$$\begin{pmatrix} \tilde{\delta}(g_1) \\ \vdots \\ \tilde{\delta}(g_k) \end{pmatrix} - \begin{pmatrix} h_{11} & \cdots & h_{1k} \\ \vdots & & \vdots \\ h_{k1} & \cdots & h_{kk} \end{pmatrix} \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}.$$

(3) Finally, let us describe the bracket $T^1 \times T^1 \to T^2$: If $g^{(1)}$ and $g^{(2)}$ are elements of $T^1(B/A)$ represented as above and $r_1, \ldots, r_k$ form a relation of $f_1, \ldots, f_k$, then we can find $r_1^{(1)}, \ldots, r_k^{(1)}$ such that $\sum_i r_i^{(1)} f_i + \sum_i r_i g_i^{(1)} = 0$, and similarly we find $r_i^{(2)}$ such that $\sum_i r_i^{(2)} f_i + \sum_i r_i g_i^{(2)} = 0$. We obtain the element $[g^{(1)}, g^{(2)}] \in T^2(B/A)$ as class of the morphism defined by $\bar{r} \mapsto \sum_i (r_i^{(2)} g_i^{(1)} + r_i^{(1)} g_i^{(2)})$.

1.1.3 Cotangent cohomology of a mapping

Aside from associating cotangent cohomology modules to an analytic A-algebra B, i. e. being given an A-module structure on the analytic algebra B by means of some morphism $f : A \to B$, one can also define cohomology functors $T^i(f, -)$ corresponding to the morphism itself. In particular, this gives rise to the so-called Kodaira-Spencer sequence which will be of central importance in the next chapter. In this section we mainly follow the treatment in [Fle78] and [Pal90b].

Definition 1.1.20. Let A, B and C be analytic algebras such that B and C are A-algebras, and let $f : B \to C$ be an A-algebra homomorphism. Choose resolvents R (resp. S) of B (resp. C) over A that are compatible with f, i. e. there is a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & C \end{array}$$

of DG-algebras over A.

(1) Define

$$\mathbb{L}^{\bullet}_{f} := (\mathbb{L}^{\bullet}_{B/A}, \mathbb{L}^{\bullet}_{C/A}).$$

Formally, this is a complex in the derived category $D^{-}(\mathcal{C}_{f:B\to C})$ of the following abelian category $\mathcal{C}_{f:B\to C}$:

i) Its objects are triples (M_1, M_2, h) of a B-module M_1, a C-module M_2 and a homomorphism $h : M_1 \to M_2$ over $f : B \to C$;

ii) $\mathrm{Hom}_{\mathcal{C}_{f:B\to C}}((M_1, M_2, h), (M_1', M_2', h'))$ consists of all pairs (g_1, g_2) of a B-module homomorphism $g_1 : M_1 \to M_1'$ and a C-module homomorphism $g_2 : M_2 \to M_2'$ such that $g_2 \circ h = h' \circ g_1$.

(2) For a B-module M we set

$$T^i(f/A, M) := \mathrm{Ext}^i_{\mathcal{C}_{f:B\to C}}(\mathbb{L}^{\bullet}_{f}, (M, M \otimes_B C)), \qquad i \geq 0.^*$$

As before, we simply write $T^i(f, M)$ if $A = \mathbb{C}$ and $T^i(f/A) := T^i(f/A, B)$.

(3) More generally, we write

$$T^i(f/A, (M_1, M_2)) := \mathrm{Ext}^i_{\mathcal{C}_{f:B\to C}}(\mathbb{L}^{\bullet}_{f}, (M_1, M_2)), \qquad i \geq 0,$$

for any object (M_1, M_2, h) in $\mathcal{C}_{f:B\to C}$.

Remark 1.1.21. Again, one can describe $T^0(f/A, -)$ more explicitly in terms of derivations: Let $f : B \to C$ be as above and let (M_1, M_2, h) be an object in $\mathcal{C}_{f:B\to C}$. Then $T^0(f/A, (M_1, M_2))$ consists of all pairs of compatible derivations $(\delta, \tilde{\delta}) \in \mathrm{Der}_A(B, M_1) \times \mathrm{Der}_A(C, M_2)$, i. e. pairs $(\delta, \tilde{\delta})$ such that

$$\begin{array}{ccc} M_1 & \xrightarrow{h} & M_2 \\ {\scriptstyle\delta}\uparrow & & \uparrow{\scriptstyle\tilde{\delta}} \\ B & \xrightarrow{f} & C \end{array}$$

commutes [Fle78, 1.19].

Remark 1.1.22. Note that, using the construction of 1.1.18, one can induce on $T^{\bullet}(f/A)$ the structure of a DG-Lie algebra [Fle78, 1.21, 1.22].

*This definition makes sense since the category $\mathcal{C}_{f:B\to C}$ has enough projective objects by [Fle78, § 1 D.].

For our purposes, the most important property of these functors is the following fact addressing the interplay between $T^i(f,-)$ and the cotangent cohomology functors introduced in the previous section:

Proposition 1.1.23. *Let $f : B \to C$ be a morphism of analytic A-algebras and (M_1, M_2, h) an object in $\mathcal{C}_{f:B\to C}$. Then there is a long exact sequence in cohomology*

$$0 \to T^0(C/B, M_2) \to T^0(f/A, (M_1, M_2)) \to T^0(B/A, M_1) \to T^1(C/B, M_2) \to \dots.$$

Proof. This is a (slight) generalization of [Fle78, 1.18]. Choose resolvents R and S of B resp. C over A as above. S can be chosen to be a free R-algebra so that $\mathbb{L}^\bullet_f = (\Omega_{R/A}\otimes_R B, \Omega_{S/A}\otimes_S C)$ is a complex of projective objects in $\mathcal{C}_{f:B\to C}$ and thus $T^i(f/A, (M_1, M_2))$ can be computed as i-th cohomology module of the complex $\mathrm{Hom}_{\mathcal{C}_{f:B\to C}}(\mathbb{L}^\bullet_f, (M_1, M_2))$. The sequence of complexes

$$0 \to \mathrm{Hom}_C(\Omega_{S/R} \otimes_S C, M_2) \to \mathrm{Hom}_{\mathcal{C}_{f:B\to C}}(\mathbb{L}^\bullet_f, (M_1, M_2)) \to \mathrm{Hom}_B(\Omega_{R/A} \otimes_R B, M_1) \to 0$$

is exact. Hence, we are done by passing to its associated long exact cohomology sequence: We can compute $T^i(C/B, M_2)$ out of the first term, the one in the middle gives $T^i(f/A, (M_1, M_2))$, and the last one yields $T^i(B/A, M_1)$. ∎

1.1.4 The cotangent complex of complex spaces

Having defined cotangent cohomology of analytic algebras and their morphisms (or, what amounts to the same, of germs of complex spaces and maps between germs), one can also define their global counterparts, associated to complex spaces and holomorphic mappings between complex spaces. We only sketch the notions and results briefly because we will only occasionally make use of them later on. For a thorough discussion we refer to [Pal76] and [Fle78].

So let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism between complex spaces. Each such morphism admits a *resolvent* $\mathcal{R}$, giving rise to the so-called *cotangent complex* $\mathbb{L}^\bullet_{X/S}$ of X over S – see [Fle78, § 2 B.-C.] for a definition and a proof of its existence. This complex does not depend on the chosen resolvent and, as in the local case discussed above, the derived functors of Hom resp. $\mathcal{H}om$ induce the cotangent cohomology modules of X over S:

Definition 1.1.24. Let $f : X \to S$ be a morphism of complex spaces and $\mathbb{L}^\bullet_{X/S}$ the cotangent complex of X over S. The *cotangent cohomology* functors of X over S are defined as

(1) $T^i(X/S, \mathcal{M}) := \mathrm{Ext}^i_{\mathcal{O}_X}(\mathbb{L}^\bullet_{X/S}, \mathcal{M})$ resp.

(2) $\mathcal{T}^i(X/S, \mathcal{M}) := \mathcal{E}\mathcal{X}\mathcal{T}^i_{\mathcal{O}_X}(\mathbb{L}^\bullet_{X/S}, \mathcal{M})$

for any $\mathcal{O}_X$-module $\mathcal{M}$.

These functors have properties analogous to those stated in Proposition 1.1.10, Proposition 1.1.12 and Proposition 1.1.13. Furthermore one has the following statement, showing that this cohomology is the correct 'globalized' version of cotangent cohomology for analytic algebras and germs:

Proposition 1.1.25.
(1) For any $\mathcal{O}_X$-module $\mathcal{M}$, the sheaves $\mathcal{T}^i(X/S, \mathcal{M})$ are coherent $\mathcal{O}_X$-modules.

(2) The stalk of $\mathbb{L}^{\bullet}_{X/S}$ at any point $x \in X$ is a cotangent complex for $\mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x}$.

Proof. [Fle78, 2.21]. ∎

Finally, one can also define a cotangent complex for a morphism $f : X \to S$ over a base space Z. This gives rise to functors $T^i(f/Z, -)$, and we end up with a cohomology sequence analogous to the one in Proposition 1.1.23:

Proposition 1.1.26. *If $f : X \to S$ is a morphism of complex spaces over the complex space Z and $\mathcal{N}$ is an $\mathcal{O}_S$-module, there is a long exact sequence*

$$0 \to T^0(X/S, f^*(\mathcal{N})) \to T^0(f/Z, \mathcal{N}) \to T^0(S/Z, \mathcal{N}) \to T^1(X/S, f^*(\mathcal{N})) \to \dots.$$

Proof. [Fle78, 3.4]. ∎

We omit a precise definition of $T^i(f/Z, -)$ and just note that for $i = 0$ one obtains, as before, the set of all compatible derivations in

$$\mathrm{Der}_{\mathcal{O}_Z}(\mathcal{O}_X, f^*(\mathcal{N})) \times \mathrm{Der}_{\mathcal{O}_Z}(\mathcal{O}_S, \mathcal{N}),$$

and $T^i(f/Z, -)$ coincides with $T^i(X/Z)$ in case $X = S$ and $f = id_X$.

1.2 Deformation theory

The second part of this first chapter is intended to give a review of the concepts and results of general deformation theory that we will rely upon later on. After introducing the abstract formalisms, we summarize the main results on deformations of singularities. In the subsequent section we will then discuss the functors of deformations with section and singular section in some more detail.

Let us fix some more notations: By (Gan) we denote the category of germs of complex spaces and by $(Analg)$ the (equivalent) category of analytic algebras.

1.2.1 Deformation functors

Definition 1.2.1. Let $f : Y_0 \to X_0$ be a morphism of germs of complex spaces.

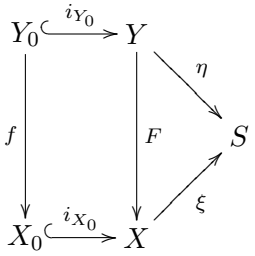
(1) A *deformation of* X_0 over the base space S consists of a flat morphism $\xi : X \to S$ in (Gan), together with an isomorphism $i_{X_0} : X_0 \xrightarrow{\sim} \xi^{-1}(0)$.

If $\xi' : X' \to S'$ is another deformation of X_0, with $i'_{X_0} : X_0 \simeq \xi'^{-1}(0)$, then a morphism of deformations is given by a morphism $\varphi : S' \to S$ of the base germs together with an isomorphism $X' \simeq X \times_S S'$, such that the induced diagram

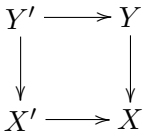
$$\begin{array}{ccc} & X_0 & \\ {\scriptstyle i'_{X_0}}\swarrow^{\sim} & & \searrow^{\sim}{\scriptstyle i_{X_0}} \\ \xi'^{-1}(0) & \longrightarrow & \xi^{-1}(0) \end{array}$$

commutes. In particular, for any morphism $\varphi : S' \to S$ the *pull-back* of ξ by φ is defined by $X \times_S S' \to S'$ and denoted as $\varphi^*(\xi)$.

(2) A *deformation of the morphism* $f : Y_0 \to X_0$ with base S consists of deformations $\xi : X \to S$ and $\eta : Y \to S$ of X_0 and Y_0, respectively, and a morphism $F : Y \to X$, such that the induced diagram



commutes. If $F' : X' \to Y'$ (and a whole diagram with primes as above) defines another deformation of f, then a morphism of deformations of f consists of morphisms $X' \to X$ and $Y' \to Y$ of deformations of X_0 and Y_0, respectively, such that the diagram



commutes. In particular, $X' \to X$ and $Y' \to Y$ are both isomorphisms.

(3) If, in the definition above, we only allow $X := X_0 \times S$, we obtain *relative deformations of* Y_0 *over* X_0 with base S; and similarly, taking $Y := Y_0 \times S$ we get a *relative deformation of* X_0 *under* Y_0.

(4) An *unfolding* of f is a deformation of f, where X_0 and Y_0 are both only deformed trivially, i. e. $X \simeq X_0 \times S$ and $Y \simeq Y_0 \times S$, so only f varies and X_0, Y_0 are both kept fixed. In this situation, the flatness condition is of course empty.

Remark 1.2.2. Any morphism of deformations over the identity map $S \to S$ is an isomorphism, as follows easily from the Inverse Function Theorem (cf. [dJP00, Theorem 10.2.10], for instance).

These notions give rise to the following contravariant *deformation functors* $(Gan) \to (Sets)$ associated to the morphism $f : Y_0 \to X_0$, see [Buc81]:

Definition 1.2.3. Let $f : Y_0 \to X_0$ be a morphism of germs of complex space, and S a further germ. We define five contravariant *deformation functors* as follows:

(1) $Def_{X_0}(S)$ is the set of isomorphism classes of deformations of X_0 with base S,

(2) $Def_{Y_0/X_0}(S)$ is the set of isomorphism classes of relative deformations of Y_0 over X_0 with base S,

(3) $Def_{Y_0\setminus X_0}(S)$ is the set of isomorphism classes of relative deformations of X_0 under Y_0 with base S,

(4) $Def_f(S)$ is the set of isomorphism classes of deformations of the morphism f with base S,

(5) $Unf_f(S)$ is the set of isomorphism classes of unfoldings of f with base S.

Analogously, one obtains covariant functors $(Analg) \to (Sets)$, which, by abuse of notation, we will denote by the same symbols.

For each of these functors, $Def_\bullet(D)$ is called its *tangent space*, where $D := (\{0\}, \mathbb{C}[\varepsilon])$ stands for the *double point* with local algebra $\mathbb{C}[\varepsilon] := \mathbb{C}\{x\}/(x^2)$.

There are a lot of natural transformations between these functors given by the 'inclusion' and 'forgetting data' – according to [Buc81], their interplay is understood best by putting them all into an octahedron! All deformation functors defined give rise to examples of cofibred groupoids over the category $(Analg)$, a concept that we are now going to recall, following [Bin80], [Sti88], [Ste96]:

Definition 1.2.4. A *cofibred groupoid* over a category C is a pair (F, p), consisting of a category F and a covariant functor $p : F \to C$, such that the following lifting conditions are satisfied:

i) *Existence:* For every morphism $\Phi : R \to S$ in C and every object r in F such that $p(r) = R$, there exists an object s in F and a morphism $\varphi : r \to s$ such that $p(s) = S$ and $p(\varphi) = \Phi$:

$$\begin{array}{ccc} r & \xrightarrow{\varphi} & s \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ R & \xrightarrow{\Phi} & S. \end{array}$$

ii) *Uniqueness:* If $R \to S \to T$ is a commutative diagram in C and $r \to s$, $r \to t$ are lifts of the morphisms $R \to S$ and $R \to T$ in F, then there is a unique lift $s \to t$ of the morphism $S \to T$ such that the whole diagram commutes:

$$\begin{array}{ccccc} r & \to & s & \overset{\exists!}{\cdots\to} & t \\ \downarrow & & \downarrow & & \downarrow \\ R & \to & S & \to & T \end{array}$$

For any object R in C, the *fibre* $F(R)$ is the subcategory of F with $ob(F(R)) := \{r \in F : p(r) = R\}$ and $\mathrm{Hom}_{F(R)}(r, r') := \{\varphi \in \mathrm{Hom}_F(r, r') : p(\varphi) = id_R\}$.

It is obvious that all deformation functors $(Analg) \to (Sets)$ from above yield examples of cofibred groupoids over the category $(Analg)$. For instance, if X_0 is a germ of a complex space, we obtain a groupoid by starting with the category (Def_{X_0}) whose objects are isomorphism classes of deformations of X_0, together with the functor $p : (Def_{X_0}) \to (Analg)$ that assigns to each isomorphism class of a deformation the local ring of its base space.

In particular, these groupoids all have the property that their fibres over $\mathbb{C}$ consist of only one single element.

Definition 1.2.5. Let F be a cofibred groupoid over $(Analg)$ such that $F(\mathbb{C})$ consists of one single element. $s \in ob(F(S))$ is called

(1) *versal*, if the following lifting property is satisfied: If $r' \to r$ is a morphism in F such that the corresponding morphism $R' \to R$ in $(Analg)$ is surjective, then every morphism $s \to r$ can be lifted to a morphism $s \to r'$ such that the composition $s \to r' \to r$ equals $s \to r$;

(2) *formally versal*, if this lifting property is satisfied for any Artinian R';

(3) *(formally) semi-universal*, if it is (formally) versal and, in addition, the induced morphism $\mathfrak{m}_S/\mathfrak{m}_S^2 \to \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2$ is uniquely defined by $r' \to r$ and $s \to r$.

(4) *universal* if it is versal and any lifting morphism $s \to r'$ is unique.

Remark 1.2.6. Quite customary is also a slightly different definition of versality, which only requires that the lifting property is satisfied for $r = s \otimes \mathbb{C}$ – i. e. in terms of deformation functors: every deformation is isomorphic to the pull-back of a versal one. However, this somewhat stronger formulation will guarantee that every versal deformation is the product of a semi-universal one with a trivial factor (see [Ste96]).

1.2.2 Cotangent cohomology and semi-universal deformations

When studying these deformation problems, there are three associated sets of particular importance:

- the respective infinitesimal automorphisms,
- the infinitesimal deformations, i. e. the tangent space of the functor under consideration, and
- the set of obstructions for lifting infinitesimal deformations to deformations of higher order.

It turns out that the cotangent cohomology modules introduced in the previous section contain all these information:

Theorem 1.2.7. *Let $f : Y_0 \to X_0$ be a morphism of germs of complex spaces.*

(1) Let $\xi : X \to D$ be a deformation of X_0 over the double point D.

- *The group of automorphisms of X that induce the identity on D is isomorphic to $T^0(X_0)$,*
- *$T^1(X_0)$ is isomorphic to $Def_{X_0}(D)$, the set of isomorphism classes of deformations over the double point,*
- *If $0 \to J \to \mathcal{O}_{D'} \to \mathcal{O}_D \to 0$ is a simple extension*, then there exists a so-called obstruction element in $T^2(X_0)$ that vanishes if and only if the deformation ξ can be lifted to a deformation over D'.*

(2) The analogue objects for the functor Def_{Y_0/X_0} are $T^i(Y_0/X_0)$ for $i = 0, 1, 2$.

(3) If $Y_0 \hookrightarrow X_0$ is a closed subspace such that $\mathcal{O}_{Y_0} \simeq \mathcal{O}_{X_0}/I$, then the correct objects for $Def_{Y_0 \setminus X_0}$ are given by $T^i(X_0, I)$.

(4) For Def_f, the corresponding objects are $T^i(f)$, $i = 0, 1, 2$.

Proof. [Buc81, section 2.4] resp. [Ill71]. ∎

*i. e., $\mathfrak{m}_{D'} J = 0$ and $\dim_{\mathbb{C}} J = 1$.

A first answer to the question whether semi-universal deformations of singularities exist is given by the classical treatment [Sch68] of Schlessinger, where some relatively weak conditions are formulated that imply the existence of a *formally* versal object for a given functor as above. In particular, they are satisfied for the deformation functor of any isolated singularity X_0. The next step then consists in approximating the formally versal deformation by an analytic one and showing that it is in fact versal itself. This is much harder and has first been proved by Grauert in [Gra72]:

Theorem 1.2.8. *Let X_0 be a singularity such that $\dim_{\mathbb{C}} T^1(X_0) < \infty$, e. g. X_0 is an isolated singularity. Then there exists a semi-universal deformation of X_0.*

The main technical tool in its proof is the Grauert Approximation Theorem – for a detailed discussion and a proof of this statement we refer to [dJP00], where it is also shown how one uses it to deduce the above theorem.

Remark 1.2.9.

(1) General conditions which ensure that formally versal objects are already versal themselves are formulated in [Bin80]. Aside from deformations of isolated singularities, they are satisfied for deformations of compact complex spaces, of coherent module sheaves and of principal bundles on compact complex spaces, to name some examples.

(2) Its is proved in [Fle81, section 5] that, using the definition of versality from above, the existence of a versal deformation also implies the existence of a semi-universal one, and the base ring $\mathcal{O}_T$ of any versal deformation is isomorphic to a free convergent power series ring $\mathcal{O}_S\{z_1, \ldots, z_r\}$, where $\mathcal{O}_S$ is the base ring of a semi-universal deformation.

Remark 1.2.10. These constructions guaranteeing the existence of semi-universal deformations of isolated singularities can actually be turned into an algorithm. Let X_0 be an isolated singularity, defined by $f = (f_1, \ldots, f_k) \subseteq \mathcal{O}_n$. First of all one needs the modules $T^1(X_0)$ and $T^2(X_0)$, which can be computed as described in Remark 1.1.15. So suppose $g^{(1)}, \ldots, g^{(\tau)} \in \mathcal{O}_n^k$ represent a $\mathbb{C}$-vector space basis of $T^1(X_0)$. Then

$$f + \sum_{i=1}^{\tau} s_i g^{(i)}$$

defines a deformation of X_0 over $\mathbb{C}\{s_1, \ldots, s_\tau\}/(s)^2$. Now suppose we are given a deformation $X_A \to A$ over an Artinian space A with $\mathcal{O}_A = \mathbb{C}\{s\}/J$. To the small extension

$$0 \to J/(s)J \to \mathbb{C}\{s\}/(s)J \to \mathcal{O}_A \to 0$$

one associates an obstruction element in

$$T^2(X_0) \otimes_{\mathbb{C}} J/(s)J \simeq \mathrm{Hom}_{\mathbb{C}}(T^2(X_0)^*, J/(s)J).$$

If $I \bmod (s)J$ denotes the image of the corresponding map, then I is the minimal ideal such that one can extend the deformation $X_A \to A$ to a deformation $X_{A'} \to A'$, where $\mathcal{O}_{A'} = \mathbb{C}\{s\}/I$. Iterating this procedure one obtains k-jets of a semi-universal deformation of X_0. A detailed account of this method, together with a description of the resulting implementation in the SINGULAR-library `deform.lib` [Mar02b] is contained in [Mar99]. Of course, this process needs not terminate, but nevertheless for a lot of interesting examples it actually does. In

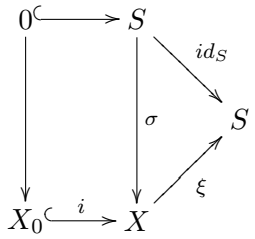
particular, if X_0 is *unobstructed*, i. e. $T^2(X_0) = 0$, then a semi-universal deformation of X_0 is defined by

$$f + \sum_{i=1}^{\tau} s_i g^{(i)} \in \mathbb{C}\{x, s\}^k.$$

A different approach computes a local flattener inside a semi-universal *unfolding* of the map germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$ defined by $f_1, \ldots, f_k$, cf. [Tei77], [Hau83]. The problem with this construction is that it involves the vector space $K_f^1 := (\mathbb{C}\{x\}/(f))^k/\overline{J(f)}$ which, in general, is not finite-dimensional (here $\overline{J(f)}$ denotes the module generated by the classes of the partial derivatives of $f_1, \ldots, f_k$).

1.3 Deformations with section and singular section

We now investigate one important special case of deformations of a morphism $Y_0 \to X_0$: namely, the case where Y_0 is trivial, i. e. only consists of the reduced point 0. Deformations of the embedding $0 \hookrightarrow X_0$ give rise to a diagram



Thus, deformations of $0 \hookrightarrow X_0$ are deformations of X_0, together with a section σ of the morphism $\xi : X \to S$. This gives rise to the next definition:

Definition 1.3.1. Let X_0 be a germ of a complex space. A *deformation with section* of X_0 over a germ S is a deformation $\xi : X \to S$ of the embedding morphism $0 \hookrightarrow X_0$. We write shortly $Def^s_{X_0}$ for the functor $Def_{0 \hookrightarrow X_0}$ and denote the deformation by (ξ, σ) if we want to mention the given section σ of ξ explicitly.

By abuse of notation, we denote by $Def^s_{X_0}$ the (covariant) functor $(Analg) \to (Sets)$ of deformations with section of the analytic algebra $\mathcal{O}_{X_0}$, too.

Of course, since the simple point $\{0\}$ can only be deformed trivially, we could consider this deformation functor as the functor of relative deformations of X_0 under 0 as well. In particular this would give the same results concerning its cotangent cohomology which will be formulated next.

1.3.1 Cotangent cohomology of $Def^s_{X_0}$

The interpretation of deformations with section of X_0 as deformations of the morphism $0 \hookrightarrow X_0$, which in turn corresponds to the residue map $\mathcal{O}_{X_0} \twoheadrightarrow \mathbb{C}$, gives rise to the fact that the corresponding cotangent cohomology coincides with the cotangent cohomology of $(X_0, 0)$ with values in $\mathfrak{m}_{X_0}$, the maximal ideal of $\mathcal{O}_{X_0}$:

Proposition 1.3.2. *Let X_0 be a germ of a complex space, minimally embedded in $(\mathbb{C}^n, 0)$ and defined by an ideal $I = (f_1, \ldots, f_k) \subseteq \mathfrak{m}^2_{\mathcal{O}_n}$. Then*

$$T^i(0 \hookrightarrow X_0) = T^i(X_0, \mathfrak{m}_{X_0}), \qquad i \geq 0.$$

In particular we have:

(1) $T^0(X_0, \mathfrak{m}_{X_0}) \simeq \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X_0}, \mathfrak{m}_{X_0})$.

(2) $T^1(X_0, \mathfrak{m}_{X_0})$ *is the cokernel of the map*

$$\begin{aligned} \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_n, \mathfrak{m}_{X_0}) &\to \mathrm{Hom}_{\mathcal{O}_{X_0}}(I/I^2, \mathfrak{m}_{X_0}) \\ \delta &\mapsto (g \bmod I^2 \mapsto \delta(g)). \end{aligned}$$

If $(X_0, 0)$ is a complete intersection and $f_1, \ldots, f_k$ form a regular sequence, then

$$T^1(X_0, \mathfrak{m}_{X_0}) \simeq \mathfrak{m}_{X_0}^{\oplus k} / (\mathfrak{m}_{X_0} \overline{J(f)}),$$

where $\overline{J(f)}$ is the module generated by the columns of the Jacobian matrix

$$\begin{pmatrix} \overline{\frac{\partial f_1}{x_1}} & \cdots & \overline{\frac{\partial f_1}{x_n}} \\ \vdots & & \vdots \\ \overline{\frac{\partial f_k}{x_1}} & \cdots & \overline{\frac{\partial f_k}{x_n}} \end{pmatrix}$$

of $(f_1, \ldots, f_k)$ modulo I.

(3) Complete intersections are unobstructed for $Def^s_{X_0}$, i. e. $T^2(X_0, \mathfrak{m}_{X_0}) = 0$.

Proof. Let r denote the residue map $\mathcal{O}_{X_0} \twoheadrightarrow \mathbb{C}$. The cotangent complex of $\mathbb{C}$ being trivial we get

$$\mathrm{Hom}_{\mathcal{C}_{0 \hookrightarrow X_0}}(\mathbb{L}^\bullet_r, (\mathbb{C}, \mathcal{O}_{X_0})) \simeq \mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathbb{L}^\bullet_{\mathcal{O}_{X_0/\mathbb{C}}}, \mathfrak{m}_{X_0}),$$

where $\mathcal{C}_{0 \hookrightarrow X_0}$ denotes the category described in section 1.1.3. Thus

$$\begin{aligned} T^i(0 \hookrightarrow X_0) &= \mathrm{Ext}^i_{\mathcal{C}_{0 \hookrightarrow X_0}}(\mathbb{L}^\bullet_r, \mathbb{C}) \\ &\simeq \mathrm{Ext}^i_{\mathcal{O}_{X_0}}(\mathbb{L}^\bullet_{\mathcal{O}_{X_0/\mathbb{C}}}, \mathfrak{m}_{X_0}) = T^i(X_0, \mathfrak{m}_{X_0}). \end{aligned}$$

Note that, alternatively, one can derive this computation from statement (3) in Theorem 1.2.7.

(1) and (3) are now a direct consequence of Proposition 1.1.10. Property (2) follows from Proposition 1.1.14, noting that $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_n, \mathfrak{m}_{X_0})$ is generated by the classes of $\left\{ x_i \frac{\partial}{\partial x_j} \right\}_{i,j}$ and that $\mathrm{Hom}_{\mathcal{O}_{X_0}}((f)/(f^2), \mathfrak{m}_{X_0}) \simeq \mathfrak{m}_{X_0}^{\oplus k}$ for complete intersections. ∎

Remark 1.3.3. In particular, as embedded deformation, every infinitesimal deformation of $X_0 \subseteq (\mathbb{C}^n, 0)$ given by $f_1, \ldots, f_k \in \mathcal{O}_n$ is defined by $f_i + \varepsilon g_i \in \mathcal{O}_n \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2)$, $i = 1, \ldots, k$, where the g_i are elements of the maximal ideal of $\mathcal{O}_n$. More generally, by a suitable coordinate change we may assume that (up to isomorphism) every deformation with section of X_0 over the base S is given by $F_1, \ldots, F_k \in (x)\mathcal{O}_S\{x\}$, and σ is the zero section.

The following basic facts relating the cotangent cohomology of Def_{X_0} and $Def^s_{X_0}$ for isolated singularities will be very useful later on:

Proposition 1.3.4. *Let X_0 be an isolated singularity. Then*

(1) $T^0(X_0, \mathfrak{m}_{X_0}) \simeq T^0(X_0)$.

(2) If, in addition, X_0 is a complete intersection, then

$$\tau^s(X_0) := \dim_{\mathbb{C}} T^1(X_0, \mathfrak{m}_{X_0}) = \tau(X_0) + \dim(X_0).$$

We call $\tau^s(X_0)$ the Tjurina number with section *of the germ X_0, in analogy to the usual Tjurina number $\tau(X_0) = \dim_{\mathbb{C}} T^1(X_0)$ of a singularity.*

Proof. By [SW77, (1.1)] we have that $Im(\delta) \subseteq \mathfrak{m}_{X_0}$ for any $\delta \in \mathrm{Der}_{\mathbb{C}}(X_0)$, hence

$$T^0(X_0, \mathfrak{m}_{X_0}) \simeq \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X_0}, \mathfrak{m}_{X_0}) = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X_0}) \simeq T^0(X_0),$$

proving (1). Now, using Proposition 1.3.2, we obtain the long exact cohomology sequence

$$0 \to T^0(X_0, \mathfrak{m}_{X_0}) \to T^0(X_0) \to T^0(X_0, \mathbb{C}) \to T^1(X_0, \mathfrak{m}_{X_0}) \to T^1(X_0) \to T^1(X_0, \mathbb{C}) \to 0$$

derived from the short exact sequence $0 \to \mathfrak{m}_{X_0} \to \mathcal{O}_{X_0} \to \mathbb{C} \to 0$ and using the fact that $T^2(X_0, \mathfrak{m}_{X_0}) = 0$. By part (1) its right hand part gives the exact sequence of $\mathbb{C}$-vector spaces

$$0 \to T^0(X_0, \mathbb{C}) \to T^1(X_0, \mathfrak{m}_{X_0}) \to T^1(X_0) \to T^1(X_0, \mathbb{C}) \to 0. \qquad (*)$$

Now let

$$n := \mathrm{e.dim}(X_0) = \dim_{\mathbb{C}}(\mathfrak{m}_{X_0}/\mathfrak{m}^2_{X_0})^* = \dim_{\mathbb{C}} T^0(X_0, \mathbb{C}),$$

and let $f_1, \ldots, f_k \in \mathcal{O}_n$ be a regular sequence generating the ideal I defining X_0, so $\dim(X_0) = n - k$. We may assume that $f_i \in (x)^2$ for all i. Then $\mathrm{Hom}_{\mathcal{O}_{X_0}}(I/I^2, \mathbb{C}) \simeq \mathbb{C}^k$ since X_0 is a complete intersection, and $T^1(X_0, \mathbb{C}) \simeq \mathrm{Hom}_{\mathcal{O}_{X_0}}(I/I^2, \mathbb{C})$ because it is the cokernel of $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_n, \mathbb{C}) \to \mathrm{Hom}_{\mathcal{O}_{X_0}}(I/I^2, \mathbb{C})$, which is the zero map. Altogether we obtain from $(*)$:

$$\begin{aligned} \dim_{\mathbb{C}} T^1(X_0, \mathfrak{m}_{X_0}) &= \dim_{\mathbb{C}} T^1(X_0) + \dim_{\mathbb{C}} T^0(X_0, \mathbb{C}) - \dim_{\mathbb{C}} T^1(X_0, \mathbb{C}) \\ &= \tau(X_0) + n - k \\ &= \tau(X_0) + \dim(X_0), \end{aligned}$$

completing the proof. ∎

In particular, suppose $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ defines an isolated hypersurface singularity X_0. Then there is a short exact sequence of finite-dimensional $\mathbb{C}$-vector spaces

$$0 \to (f, J(f))/(f, \mathfrak{m}J(f)) \to \mathbb{C}\{x\}/(f, \mathfrak{m}J(f)) \to \mathbb{C}\{x\}/(f, J(f)) \to 0,$$

which implies that the first of these three vector spaces has dimension

$$(\tau^s(X_0) + 1) - \tau(X_0) = n,$$

so the partial derivatives not only generate but actually form a basis of this vector space, i. e.:

Corollary 1.3.5. *Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ define an isolated hypersurface singularity. Then $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ are linearly independent modulo $(f, \mathfrak{m}J(f))$ and thus also modulo $\mathfrak{m}J(f)$.*

1.3.2 Construction of a versal deformation with section

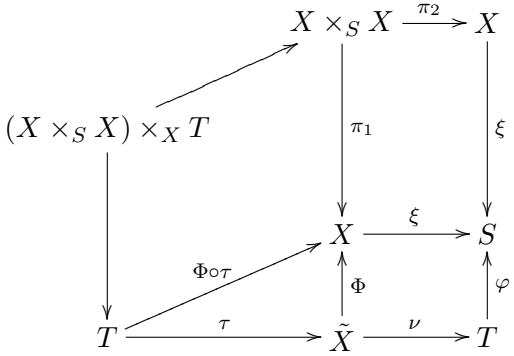
We now turn to the construction of a versal deformation with section of an isolated singularity X_0, which goes back to [Buc81].

Theorem 1.3.6. *Suppose $\xi : X \to S$ is a versal deformation of X_0. Then $\pi_1 : X \times_S X \to X$ together with the diagonal embedding $d : X \to X \times_S X$ gives a versal deformation with section of X_0.*

Proof (following [Buc81], see also [MvS01]). We verify the definition of versality, as given in Definition 1.2.5. So let $T' \hookrightarrow T$ be an injective morphism of germs and $\nu : \tilde{X} \to T$ and $\nu' : X' \to T'$ deformations of X_0 with sections τ, τ':

$$\begin{array}{ccc} T' & \hookrightarrow & T \\ \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau} \\ X' & \hookrightarrow & \tilde{X} \\ \downarrow{\scriptstyle \nu'} & & \downarrow{\scriptstyle \nu} \\ T' & \hookrightarrow & T \end{array}$$

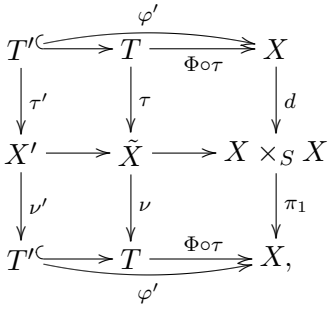
Suppose ν' is induced (as element of $Def_{X_0}(T')$, i. e. forgetting the section) from ξ by some morphism $\varphi' : T' \to S$. Versality of ξ implies the existence of some $\varphi : T \to S$ such that ν is isomorphic to the pull-back of ξ via φ, so we may assume $\tilde{X} = X \times_S T$. Denoting by Φ the projection $\tilde{X} \to X$ we obtain, by pulling back $X \times_S X \to X$ over $\Phi \circ \tau$:



From $\tilde{X} = X \times_S T$ we deduce that (up to isomorphism):

$$(X \times_S X) \times_X T \simeq X \times_S (X \times_X T) \simeq X \times_S T = \tilde{X}.$$

Under these identifications, we can identify $d \circ (\Phi \circ \tau)$ with τ. Altogether this proves that (ν, τ) is isomorphic to the pull-back of (π_1, d) via $\Phi \circ \tau$, and so we end up with the commutative diagram



proving the $Def^s_{X_0}$-versality of $X \times_S X \to X$. ∎

From the theorem we immediately deduce, by splitting of a superfluous factor, if necessary:

Corollary 1.3.7. *For every isolated singularity X_0, there exists a semi-universal deformation with section of X_0.*

Furthermore, combining this result with the calculation of $\tau^s(X_0)$ in Proposition 1.3.4, we obtain:

Corollary 1.3.8. *If X_0 is an isolated complete intersection singularity, then the construction of Theorem 1.3.6 yields a semi-universal deformation with section of X_0, when starting with a semi-universal deformation of X_0.*

Example 1.3.9 (Complete intersections). We give an explicit construction of a semi-universal deformation with section of an isolated complete intersection singularity (ICIS) in terms of a defining regular sequence $f = (f_1, \ldots, f_k) \in \mathbb{C}\{x_1, \ldots, x_n\}^k$: Take a family $\{g^{(1)}, \ldots, g^{(\tau)}\}$ of monomials (i. e. $g^{(i)} = x^{\alpha_i} e_{k(i)}$, where e_j denotes the j-th unit vector) representing a $\mathbb{C}$-vector space basis of $T^1(X_0)$. We may assume that $g^{(i)} = e_i$ for $i = 1, \ldots, k$. Then

$$\begin{aligned}\mathcal{O}_S := \mathbb{C}\{u,s\}/(F_1(u,s),\ldots,F_k(u,s)) \\ \to \mathbb{C}\{x,u,s\}/(F_1(x,s),\ldots,F_k(x,s),F_1(u,s),\ldots,F_k(u,s)) =: \mathcal{O}_X\end{aligned}$$

defines a semi-universal deformation with section of X_0, where

$$\begin{pmatrix} F_1(x,s) \\ \vdots \\ F_k(x,s) \end{pmatrix} := \begin{pmatrix} f_1(x) \\ \vdots \\ f_k(x) \end{pmatrix} + \sum_{i=1}^{\tau} s_i g^{(i)}(x)$$

and the section is given by

$$\mathcal{O}_X \to \mathcal{O}_S, \quad s_i \mapsto s_i,\ u_i \mapsto u_i,\ x_i \mapsto u_i.$$

$\mathcal{O}_S$ is regular, using the relation

$$\begin{pmatrix} s_1 \\ \vdots \\ s_k \end{pmatrix} = -\begin{pmatrix} f_1(u) \\ \vdots \\ f_k(u) \end{pmatrix} - \sum_{i=k+1}^{\tau} s_i g^{(i)}(u)$$

over $\mathcal{O}_S$ resp. $\mathcal{O}_X$ we can eliminate $s_1, \ldots, s_k$ from the equations and end up with

$$\mathbb{C}\{u, s_{k+1}, \ldots, s_\tau\} \to \mathbb{C}\{x, u, s_{k+1}, \ldots, s_\tau\}/(\tilde{F}_1(x,u,s), \ldots, \tilde{F}_k(x,u,s)),$$

where

$$\begin{pmatrix} \tilde{F}_1(x,u,s) \\ \vdots \\ \tilde{F}_k(x,u,s) \end{pmatrix} := \begin{pmatrix} f_1(x)-f_1(u) \\ \vdots \\ f_k(x)-f_k(u) \end{pmatrix} + \sum_{i=k+1}^{\tau} s_i(g^{(i)}(x) - g^{(i)}(u)).$$

Remark 1.3.10. For arbitrary isolated singularities X_0 defined by $f_1, \ldots, f_k \in \mathcal{O}_n$ one may take a different approach to determine (at least k-jets of) a semi-universal deformation with section, adapting the procedure described in Remark 1.2.10: First compute $g^{(1)}, \ldots, g^{(\tau^s)} \in \mathcal{O}_n^k$ representing a $\mathbb{C}$-vector space basis of $T^1(X_0, \mathfrak{m}_{X_0})$. We obtain a semi-universal family of first order deformations

$$F^{(1)} := f + \sum_{i=1}^{\tau^s} s_i g^{(i)} \in \left(\mathcal{O}_n \otimes \mathbb{C}\{s\}/(s)^2\right)^k$$

over $\mathbb{C}\{s\}/(s)^2$. Then one lifts this family order by order (in s), killing the obstructions which this time lie in $T^2(X_0, \mathfrak{m}_{X_0})$.

Since complete intersections are unobstructed, it follows that in particular a semi-universal deformation with section of an isolated complete intersection singularity X_0 is given by

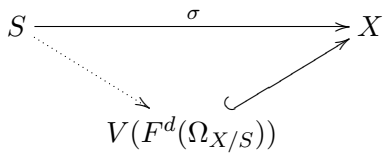
$$F := f + \sum_{i=1}^{\tau^s} s_i g^{(i)} \in \mathcal{O}_{n+\tau^s}^k.$$

1.3.3 Deformations with singular section

Having discussed the functor of deformations with section associated to a germ X_0 of a complex space, we can also define, at least for equidimensional germs, the functor of deformations with *singular section*:

Definition 1.3.11. Let X_0 be an equidimensional germ of a complex space of dimension d and $\xi : X \to S$ a flat map of germs which is equidimensional of relative dimension d. The *critical locus* of ξ is the germ $C_\xi := V(F^d(\Omega_{X/S})) \subseteq X$, where $\Omega_{X/S}$ is the module of Kähler differentials of the $\mathcal{O}_S$-module $\mathcal{O}_X$ and $F^d(\ldots)$ denotes the d-th Fitting ideal (see [Tei77], for instance).

A *deformation with singular section* of a germ X_0 of a complex space is a deformation $\xi : X \to S$ with section $\sigma : S \to X$, such that there is a factorization



Proposition 1.3.12. *Let X_0 be equidimensional of dimension d. Setting*

$$Def^{ss}_{X_0}(S) := \left\{ \begin{array}{c} \textit{isomorphism classes of deformations with singular} \\ \textit{section of } X_0 \textit{ over the base } S \end{array} \right\}$$

defines a subfunctor of $Def^s_{X_0}$.

Proof. This follows from the fact that formation of the critical space is compatible with base change, cf. [Tei77, §§ 1–2]. I. e., if $\xi : X \to S$ is a deformation with section of X_0 and $\xi' : X' \to S'$ induced via $\varphi : S' \to S$, then

$$C_{\xi'} = V(F^d(\Omega_{X'/S'})) = V(F^d(\Omega_{X/S})) \times_S S' = C_\xi \times_S S'.$$

∎

Proposition 1.3.13. *Let X_0 be equidimensional of dimension d.*

(1) A versal deformation of X_0 with respect to $Def^{ss}_{X_0}$ is obtained as follows: Let $\xi : X \to S$ be a versal deformation of X_0 (with respect to Def_{X_0}) and $X \times_S X \to X$ the versal deformation for $Def^s_{X_0}$ obtained in Theorem 1.3.6. Then the deformation induced by the base change

$$C_\xi \hookrightarrow X$$

is a versal deformation with singular section of X_0.

(2) If X factors (formally) as $X' \times Spec\ \mathbb{C}[[t_1, \ldots, t_e]]$ for some e, then a semi-universal deformation with singular section is obtained by the base change

$$V(F^{d-e}(\Omega_{X'/S'})) \hookrightarrow X'.$$

In particular, every isolated equidimensional singularity admits a semi-universal deformation with singular section.

Proof. [Buc81, 2.5.1]. ∎

Remark 1.3.14. If $f \in \mathcal{O}_n$ is a power series defining a reduced isolated hypersurface singularity, then one can easily construct a semi-universal deformation with singular section of $X_0 := (V(f), 0)$ by setting $F := f + \sum_i s_i g_i \in \mathcal{O}_n \otimes \mathbb{C}\{s\}$, where $\{g_i\}_i$ represent a $\mathbb{C}$-basis of $\mathfrak{m}_{X_0} \cdot T^1(X_0, \mathfrak{m}_{X_0})$. This is a straightforward consequence of Remark 1.3.10.

Chapter 2

Modular deformations

Roughly speaking, a geometric moduli problem can be formulated as follows (for a precise definition and a thorough discussion, see e. g. [Pop77] or [New78]): Suppose we are given a collection $\mathcal{A}$ of objects (in algebraic or analytic geometry), together with an equivalence relation on $\mathcal{A}$ and a notion of a family of objects in $\mathcal{A}$ parametrized by objects S from an appropriate category $\mathcal{S}$, satisfying suitable compatibility conditions. We thus can define a functor $F_{\mathcal{A}} : \mathcal{S} \to (Sets)$ by associating to $S \in \mathcal{S}$ the set of equivalence classes of families parametrized by S. A (fine) moduli space is then defined as a pair (M, Φ), where M is an object in $\mathcal{S}$ and $\Phi : F_{\mathcal{A}} \to \mathrm{Hom}_{\mathcal{S}}(-, M)$ is a natural transformation representing $F_{\mathcal{A}}$. Hence (in case the objects of $\mathcal{S}$ are actually sets) we can think of the elements of M as representatives of the equivalence classes of objects in $\mathcal{A}$, and every family in $\mathcal{A}$ parametrized by $S \in \mathcal{S}$ is induced by a unique morphism $S \to M$.

Now let us turn to the geometric objects that we study in this work: germs of complex spaces (with isolated singular point 0). It is well-known that fine moduli spaces in the above sense do not exist, which is a consequence of the fact that, in general, semi-universal deformations are not universal, i. e. Def_{X_0} is not representable. The simplest example of this fact is already given by the singularity defined by $f := x^2 \in \mathbb{C}\{x\}$: We obtain a semi-universal deformation of $(V(f), 0) \subseteq (\mathbb{C}, 0)$ from $F := x^2 + t \in \mathbb{C}\{x, t\}$. The homomorphisms

$$\begin{aligned} \varphi, \psi : \mathbb{C}\{t\} &\to \mathbb{C}\{z\} \\ \varphi(t) &:= z \\ \psi(t) &:= z + z^2 \end{aligned}$$

define distinct morphisms $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$, but the induced deformations, given by $F_\varphi = x^2 + z$ and $F_\psi = x^2 + z + z^2$, respectively, are isomorphic. To see this, just define $\Xi : \mathbb{C}\{x, z\} \to \mathbb{C}\{x, z\}$ by $z \mapsto z$ and $x \mapsto x\sqrt{1+z}$, then $\Xi(x^2 + z + z^2) = (1 + z)(x^2 + z)$.

In general, any isolated complete intersection singularity gives an example of a semi-universal deformation where different inducing morphisms can produce isomorphic deformations. One feasible approach to avoid this problem is to look out for a coarse moduli space instead of a fine one and restricting the particular problem by fixing invariants of the singularities under consideration, e. g. fixing the Milnor number – see [GP94], [GHP97] or [FK00], for example.

A different idea goes back to V. P. Palamodov's work [Pal78]: Inside the base space of a semi-universal deformation, single out a maximal subspace M such that uniqueness of the inducing morphisms is satisfied at least for all families induced by M. He calls such a subspace

the *modular stratum* and proves its existence for the functor of deformations of a compact complex space. He extends this result to analytic polyhedra in [Pal94]. Other generalizations, e. g. to deformations of graded spaces, have been obtained by S. Kosarew and H. Stieber, cf. [KS90], [Sti90], [Kos91]. For a recent overview of the results on modular spaces in complex analytic geometry we refer to [Pal02]. The similar notion of *prorepresenting substratum* has been introduced and investigated by O. A. Laudal and G. Pfister in [LP88], mainly focusing on the formal deformation theory of affine schemes.

In this chapter, we study the modular stratum inside the base space of a semi-universal deformation of an isolated singularity. We formulate and prove several equivalent criteria that guarantee the existence of the modular stratum of any isolated singularity. For the functor Def_{X_0} they are more or less known, our contribution is to derive them in a mostly self-contained presentation and to give a generalization to the functor of deformations with section. Having done this, we explain an interpretation in terms of flatness of a variant of the first relative cotangent cohomology module, which can then be turned into an algorithm to compute the modular stratum of a singularity. This will then be the content of chapter three. Besides, we show that that the modular stratum of an unobstructed singularity (with respect to the functor Def_{X_0} resp. $Def^s_{X_0}$) coincides with the flattening stratum of its relative T^1 itself, hence for a semi-universal-deformation $X \to S$ of such a singularity $M \subseteq S$ is modular if and only if its restriction to M is $\mathcal{O}_M$-free. This has been proved for complete intersections and reduced space curves in [Mar02a] and [Mar03], here we can give a generalization to arbitrary unobstructed singularities.

2.1 Modular subspaces and the Kodaira-Spencer sequence

2.1.1 The Kodaira-Spencer sequence of a deformation

We begin this section by introducing the so-called Kodaira-Spencer sequence of a deformation. In this form, it first appeared in [Pal90b], including the sequence of Proposition 1.1.23 obtained in [Fle78]. We present it here in a slightly generalized version, adapted to deformations with sections as well.

Proposition 2.1.1.

(1) Let X_0 be a germ of a complex space and $\xi : X \to S$ a deformation of X_0. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & T^0(X/S) & \longrightarrow & T^0(\xi) & \longrightarrow & T^0(S) & \xrightarrow{\Theta_\xi} & T^1(X/S) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev'} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T^0(X_0) & \longrightarrow & T^0(\xi,(\mathbb{C},\mathcal{O}_{X_0})) & \longrightarrow & T^0(S,\mathbb{C}) & \xrightarrow{\theta_\xi} & T^1(X_0) & \longrightarrow & \cdots . \end{array} \tag{2.1.1}$$

(2) If $\xi : X \to S$ is a deformation of X_0 with section σ, then we obtain the following

diagram, again with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^0(X/S, J_\sigma) & \longrightarrow & T^0(\xi, (\mathcal{O}_S, J_\sigma)) & \longrightarrow & T^0(S) & \xrightarrow{\Theta^s_\xi} & T^1(X/S, J_\sigma) \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev'} & & \downarrow & & \downarrow \\
0 & \longrightarrow & T^0(X_0, \mathfrak{m}_{X_0}) & \longrightarrow & T^0(\xi, (\mathbb{C}, \mathfrak{m}_{X_0})) & \longrightarrow & T^0(S, \mathbb{C}) & \xrightarrow{\theta^s_\xi} & T^1(X_0, \mathfrak{m}_{X_0}) \longrightarrow \cdots,
\end{array} \tag{2.1.2}$$

where $\mathfrak{m}_{X_0}$ denotes the maximal ideal of $\mathcal{O}_{X_0}$ and $J_\sigma := Ker(\sigma^*) \subseteq \mathcal{O}_X$.

Proof. By Proposition 1.1.23, there is a long exact sequence in cohomology

$$0 \to T^0(X/S, M_2) \to T^0(\xi, (M_1, M_2)) \to T^0(S, M_1) \to T^1(X/S, M_2) \to \ldots,$$

for every triple (M_1, M_2, h), where M_1 is an $\mathcal{O}_S$-module, M_2 is an $\mathcal{O}_X$-module and $h : M_1 \to M_2$ a morphism (of $\mathcal{O}_S$-modules) over $\xi^* : \mathcal{O}_S \to \mathcal{O}_X$.

The rows of (1) are obtained by taking $M_1 := \mathcal{O}_S$ and $M_2 := \mathcal{O}_X$ for the upper row and $M_1 := \mathbb{C} = \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{O}_S/\mathfrak{m}_S$, $M_2 := \mathcal{O}_{X_0} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_S/\mathfrak{m}_S$ for the lower row, and the commutative squares then result from the functoriality of cotangent cohomology.

For part (2), note that the zero map $\mathcal{O}_S \to J_\sigma$ gives a morphism over ξ^*, hence we obtain the diagram (2.1.2) by taking coefficients in $(\mathcal{O}_S, J_\sigma)$ and $(\mathbb{C}, \mathfrak{m}_{X_0}) = (\mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{O}_S/\mathfrak{m}_S, J_\sigma \otimes_{\mathcal{O}_S} \mathcal{O}_S/\mathfrak{m}_S)$, respectively. ∎

Definition 2.1.2. Let $\xi : X \to S$ be a deformation of X_0 (with section σ). The sequence (2.1.1) resp. (2.1.2) is called the *Kodaira-Spencer sequence* of the deformation ξ. θ_ξ (resp. θ^s_ξ) is called its *Kodaira-Spencer map* and Θ_ξ (resp. Θ^s_ξ) is the *relative Kodaira-Spencer map.*

Remark 2.1.3.

(1) The first two maps in the Kodaira-Spencer sequence have an obvious interpretation: $T^0(X/S) \to T^0(\xi)$ is the inclusion $\delta \mapsto (0, \delta)$ (considering an $\mathcal{O}_S$-derivation on $\mathcal{O}_X$ now only as a $\mathbb{C}$-derivation), and $T^0(\xi) \to T^0(S)$ is the projection to the first factor. θ_ξ and Θ_ξ have the following explicit description: $T^0(S, \mathbb{C})$ can be identified with $\mathrm{Hom}(D, S)$, where D is the double point with $\mathcal{O}_D = \mathbb{C}[\varepsilon]$. Then, for $\delta \in T^0(S, \mathbb{C})$, $\theta_\xi(\delta)$ is the isomorphism class of the deformation $\delta^*(\xi)$ in $T^1(X_0) \simeq Def_{X_0}(D)$. Similarly, the elements of $T^0(S)$ correspond to morphisms $S \times D \to S$, and $\Theta_\xi(\delta)$ is the isomorphism class of the relative deformation of X over S with base space D:

$$\begin{array}{ccc}
\tilde{X} & \longrightarrow & X \\
\swarrow & & \downarrow{\scriptstyle \xi} \\
D & \overset{\delta^*(\xi)}{\downarrow} & \\
\nwarrow & & \\
S \times D & \xrightarrow{\delta} & S.
\end{array}$$

Analogously, via θ^s_ξ any element of $T^0(S, \mathbb{C})$ induces a deformation with section over the double point, and Θ^s_ξ associates to a derivation $\delta \in T^0(S)$ the induced relative infinitesimal deformation.

(2) Injectivity of the Kodaira-Spencer map θ_ξ (resp. θ^s_ξ) is equivalent to the fact that, for any deformation (with section) induced from ξ, the inducing map is unique on the tangent level. Following [Pal90b], we call a deformation with injective Kodaira-Spencer map a *monodeformation*. So, in particular, a semi-universal deformation of X_0 is a versal monodeformation.

(3) More generally, one can construct a variant of the Kodaira-Spencer sequence for the deformation functor $Def_{Y_0\setminus X_0}$: If $\xi : X \to S$ is a deformation of X_0 over Y_0, then there is a commutative diagram

$$\begin{array}{ccc} Y_0 \times S & \xrightarrow{\sigma} & X \\ & \searrow \quad \swarrow_{\xi} & \\ & S, & \end{array}$$

so if we set $J_\sigma := Ker(\sigma^* : \mathcal{O}_X \to \mathcal{O}_{Y_0\times S})$ and denote by J_0 the ideal of Y_0 in $\mathcal{O}_{X_0}$, then we obtain a sequence analogous to (2.1.2) by taking coefficients in $(\mathcal{O}_S, J_\sigma)$ and $(\mathbb{C}, J_0)$, respectively. This generalizes the case of deformations with section, where Y_0 is the reduced point $\{0\}$, and the case of (ordinary) deformations, where Y_0 is empty.

2.1.2 Modular subspaces

Definition 2.1.4. Let $\xi : X \to S$ be a deformation of the germ of a complex space X_0 (with section σ).

(1) A subspace $M \subseteq S$ is called *modular* if the following condition holds: If $\varphi : T \to M$ and $\psi : T \to S$ are morphisms such that the induced deformations $\varphi^*(\xi_{|M})$ and $\psi^*(\xi)$ with base T are isomorphic as deformations (with section), then $\varphi = \psi$.

(2) The restriction of a deformation to a modular subgerm is called a *modular deformation*.

(3) A maximal (with respect to inclusion) modular subspace in the base space of a semi-universal deformation (with section) of X_0 is called the *modular stratum* (with section) of X_0 and will be denoted by M_{X_0}.

The last definition is justified by the following lemma – and by the proof of its existence for any isolated singularity X_0 which we will give in Corollary 2.2.7 below:

Lemma 2.1.5. *Let $\xi : X \to S$ and $\xi : X' \to S'$ be semi-universal deformations of X_0 and let $M \subseteq S$, $M' \subseteq S'$ be maximal modular subspaces. Then there is a unique isomorphism $\varphi : M \xrightarrow{\sim} M'$.*

The analogous statement holds for deformations with section.

Proof. (cf. [Pal02, Proposition 2.2]) Since ξ and ξ' are both semi-universal, there exists an isomorphism $\varphi : S \to S'$ such that $\xi \simeq \varphi^*(\xi')$. In particular, by the maximality condition on M and M' this implies that φ restricts to an isomorphism of the germs M and M' (since $\varphi(M) \cup M' \subseteq S'$ is again modular by the next lemma below). Finally, its uniqueness is a consequence of the modularity of M and M'. ∎

The following basic properties are immediate from the definition, the second of them has already been used in the above proof:

Lemma 2.1.6. *Let $\xi : X \to S$ be a deformation of X_0 (with section σ).*

(1) Any subgerm of a modular germ $M \subseteq S$ is again modular.

(2) If M_1, M_2 are modular subgerms of S, then $M_1 \cup M_2 \subseteq S$ is modular, too.

(3) $M \subseteq S$ is modular if and only if the condition of the definition is satisfied for any two morphisms $T \to M$, $T \to S$ where T is an Artinian *germ.*

Proof. The first two statements are checked directly using the definition, whereas the last one is a consequence of the identity theorem for power series. ∎

Proposition 2.1.7. *Let $\xi : X \to S$ be a deformation of X_0 (with section σ). $\{0\} \subseteq S$ is a modular subspace if and only if the corresponding Kodaira-Spencer map $\theta_\xi : T^0(S, \mathbb{C}) \to T^1(X_0)$ (resp. $\theta^s_\xi : T^0(S, \mathbb{C}) \to T^1(X_0, \mathfrak{m}_{X_0})$) is injective.*

Proof. We give a proof for the functor of deformations with section – the statement for Def_{X_0} follows by 'forgetting any section' and treating all deformations as 'ordinary' ones.

So suppose θ^s_ξ is injective first. We use the description of θ^s_ξ from Remark 2.1.3. Let Z be an Artinian germ and $\varphi : Z \to S$ a morphism such that $\varphi^*(\xi)$ is isomorphic to the trivial deformation $X_0 \times Z \to Z$. Clearly, $\varphi = 0$ if $Z = \{0\}$, so we are done by showing that $\varphi = 0$ under the assumption that we have already $\varphi_{|Z'} = 0$, whenever

$$0 \to J \to \mathcal{O}_Z \to \mathcal{O}_{Z'} \to 0$$

is a simple extension. Thus we may assume that φ is given by

$$s_i \mapsto a_i + \varepsilon c_i,\ c_i \in \mathbb{C},$$

where s_i are generators of $\mathfrak{m}_S$, ε is a generator of J as $\mathbb{C}$-vector space and $s_i \mapsto a_i \bmod J$ defines $\varphi_{|Z'}$. Since this map is trivial, we can assume that $a_i = 0$ for all i. But now we end up with a map

$$\mathcal{O}_S \to \mathbb{C}[\varepsilon],\ s_i \mapsto \varepsilon c_i,$$

inducing a trivial deformation over the double point D with $\mathcal{O}_D = \mathbb{C}[\varepsilon]$, so $c_i = 0$ for all i by the injectivity of θ_ξ.

On the other hand, suppose $\{0\} \subseteq S$ is modular. If $\delta \in Ker(\theta^s_\xi)$, then δ corresponds to $\psi_\delta : D \to S$ inducing the trivial deformation with base D. Thus $\psi^*_\delta(\xi) \simeq \varphi^*(\xi_{|\{0\}})$, where $\varphi : D \to \{0\}$ is the zero map. Hence $\psi_\delta = \varphi = 0$ by the modularity of $\{0\}$, so θ^s_ξ is injective. ∎

Together with Lemma 2.1.6 (1) this implies that, if $\xi : X \to S$ is not a monodeformation, then S cannot contain a modular subspace at all. Hence, when studying modular germs, we can restrict our attention exclusively to monodeformations.

Of course, the notion of modular subspace can be defined for a general deformation groupoid. For instance, this had been done in [KS90]. In this paper there is given the following general description of modular subgerms, which we now state for the special case of the deformation functors that we are interested in. We use the following notions:

Definition 2.1.8. For a germ of a complex space S we denote by (Gan/S) the category of analytic germs over S. If $\xi : X \to S$ is a deformation of the germ X_0, then

$$\underline{\mathrm{Aut}}_\xi : (Gan/S) \to (Groups)$$

is the functor $(T \xrightarrow{\varphi} S) \mapsto \mathrm{Aut}_T(\varphi^*(\xi))$. Analogously, we define $\underline{\mathrm{Aut}}^s_\xi$ in case ξ is a deformation with section $\sigma : S \to X$.

Note that, in this situation, one has an obvious natural transformation $\underline{\mathrm{Aut}}_\xi \to \mathrm{Hom}(-, S)$.

Proposition 2.1.9. *Let $\xi : X \to S$ be a monodeformation of X_0 and $M \subseteq S$ a subspace. If $\underline{\mathrm{Aut}}_{\xi|M} \to \mathrm{Hom}(-, M)$ is smooth, then M is modular. The converse is also true provided that ξ is semi-universal.*

The analogous statement holds for a deformation with section, replacing $\underline{\mathrm{Aut}}_\xi$ by $\underline{\mathrm{Aut}}^s_\xi$.

Proof. This is proved (in much more general context) in [KS90, Theorem 3.3] – actually they prove this statement only under the assumption on ξ to be versal, too, but this condition is only used in the proof of the 'only if'-part.

So let $M \subseteq S$ be modular first. Suppose a deformation $\eta : Y \to T$ is induced from a morphism $\varphi : T \to M$ and α is an automorphism of the restricted deformation $\eta_{|T'}$, where $T' \hookrightarrow T$ is an embedding. By versality it can also be induced by some morphism $\psi : T' \to S$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
Y_{|T'} & \xrightarrow{\alpha} & Y_{|T'} & \hookrightarrow & Y & \longrightarrow & X_{|M} & \hookrightarrow & X \\
\downarrow{\scriptstyle \eta_{|T'}} & & \downarrow{\scriptstyle \eta_{|T'}} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \xi_{|M}} & & \downarrow{\scriptstyle \xi} \\
T' & = & T' & \hookrightarrow & T & \xrightarrow{\varphi} & M & \hookrightarrow & S,
\end{array}$$

$$\psi$$

Since M is modular, ψ is equal to the composition $\varphi \circ i_{T' \hookrightarrow T}$, from which we conclude that α lifts to an automorphism of η.

Conversely, suppose $\underline{\mathrm{Aut}}_{\xi|M} \to \mathrm{Hom}(-, M)$ is smooth. In order to verify that M is modular, let be given an Artinian germ T and morphisms $\varphi : T \to M$ and $\psi : T \to S$, such that the induced deformations $\varphi^*(\xi_{|M})$ and $\psi^*(\xi)$ are isomorphic deformations over T by means of some isomorphism α. By induction we may suppose that we have already found a subgerm $T_0 \subseteq T$ such that $\varphi_{|T_0} = \psi_{|T_0}$ and $\mathcal{O}_T \twoheadrightarrow \mathcal{O}_{T_0}$ is a simple extension with kernel J. Hence $\alpha_{|T_0}$ defines an automorphism of $\varphi^*(\xi_{|M})_{|T_0}$, which can be lifted to an automorphism of $\varphi^*(\xi_{|M})$. Therefore we may also assume that the restriction of α to T_0 is the identity. Thus we can define a map $\varphi^* - \psi^* : \mathcal{O}_S \to J$ which is a derivation, so this produces an algebra homomorphism $\rho : \mathcal{O}_S \to \mathbb{C}[\varepsilon]$. Let η denote the deformation over the double point D induced from ξ by ρ.

Now $Def_{X_0}(D)$ acts in this situation freely and transitive* on the subset of $Def_{X_0}(T)$ of those deformations whose restriction to T_0 is $\varphi^*(\xi_{|M})_{|T_0}$. $\varphi^*(\xi_{|M})$ and $\psi^*(\xi)$ differ by η with respect to this action.† But since they are isomorphic we conclude that η is trivial, hence $\rho = 0$ as ξ is a monodeformation. This shows that $\varphi = \psi$, so M is modular. ∎

*This follows directly from the Schlessinger conditions, cf. [Kos88, (3.4)].

†Actually, this action is part of a whole exact sequence $T^1(X_0) \to Def_{X_0}(T) \to Def_{X_0}(T_0) \to T^2(X_0)$ also including the obstructions for lifting deformations over T_0 to a deformation over T. We refer to [Pal78, section 4] for a thorough discussion.

Remark 2.1.10. In fact, for the case deformations of complex spaces, a proof of this statement is already contained in [Pal78, Theorem 5.4]. Similarly, in [LP88] what is there called the prorepresenting substratum of the formal moduli of an affine scheme X_0 (in the base of a semi-universal deformation of X_0) is introduced differently, but it is shown in §1 of that monograph that it coincides with our notion of modular subspace, and it is also proved a criterion analogous to Proposition 2.1.9.

2.2 Characterizations of modular subspaces

In this section, we will prove the first of the main results of this chapter (Theorem 2.2.4 and Theorem 2.2.5). On the one hand they guarantee the existence of the modular stratum for any isolated singularity X_0, on the other hand they will form the starting point for an algorithmic description of how it can be computed in practice.

2.2.1 Artinian and infinitesimal modularity

First of all, we introduce two more auxiliary notions, the first of them being motivated by the fact that $\{0\} \subseteq S$ is modular if and only if θ_ξ is injective, as proved in Proposition 2.1.7. A slightly different formulation of these concepts has already appeared in [Mar03].

Definition 2.2.1. A subgerm $M \subseteq S$ in the base space of a deformation $\xi : X \to S$ (with section σ) of X_0 is called

(1) *infinitesimally modular* if the restriction to M of the map

$$\tilde{\Theta}_{\xi|M} : T^0(S, \mathcal{O}_M) \to T^1(X/S)_{|M}$$

(resp. $\tilde{\Theta}^s_{\xi|M} : T^0(S, \mathcal{O}_M) \to T^1(X/S, J_\sigma)_{|M}$ if ξ is considered as deformation with section σ) is injective;

(2) *Artinian modular* if any Artinian subgerm of M is modular.

(Here and subsequently we use the notations $T^i(X/S)_{|M} := T^i(X_{|M}/M)$ etc. for the relative cotangent cohomology of the restricted deformation $\xi_{|M}$.)

Lemma 2.2.2. *Let $\xi : X \to S$ be a monodeformation of X_0 (with section σ).*

(1) For both deformation functors, modular subspaces are infinitesimally modular.

(2) Conversely, any Artinian modular subspace M of S is modular.

Proof. ad (1): Let $M \subseteq S$ be a modular subspace and suppose $\delta \in Ker(\tilde{\Theta}_{\xi|M}) \subseteq T^0(S, \mathcal{O}_M)$. Then, similarly to the description of Θ_ξ in Remark 2.1.3, δ corresponds to a morphism $\varphi_\delta : M \times D \to S$ such that the deformation $\varphi^*_\delta(\xi)$ is isomorphic to the trivial one, which can also be induced from $\xi_{|M}$ via the projection $pr_M : M \times D \to M$. By the modularity of M we conclude that $\varphi_\delta = i \circ pr_M$, hence $\delta = 0$.

ad (2): Let Z be an Artinian germ and $\varphi : Z \to M$, $\psi : Z \to S$ morphisms such that $\varphi^*(\xi_{|M}) \simeq \psi^*(\xi)$. We can factor $\varphi = i \circ \varphi_0$, where $\varphi_0 : Z \to M_0$, M_0 is Artinian and $i : M_0 \hookrightarrow M$ stands for the inclusion. Thus $\psi^*(\xi) \simeq \varphi_0^*(i^*(\xi_{|M})) \simeq \varphi_0^*(\xi_{|M_0})$. By the assumption M_0 is modular, so $\psi = \varphi_0$ and therefore ψ and φ coincide, too.

Mutatis mutandis one obtains the analogous statements for the functor $Def^s_{X_0}$. ∎

2.2.2 A criterion for modularity

The characterizations of modular subspaces presented below depend heavily on a careful analysis of the Kodaira-Spencer sequence (2.1.1) resp. (2.1.2). In particular, the following technical lemma will form a key point:

Lemma 2.2.3. *Suppose $\xi : X \to S$ is a semi-universal deformation of X_0 (with section σ). Then:*

(1) The mapping

$$ev' : T^0(\xi) \to T^0(\xi, (\mathbb{C}, \mathcal{O}_{X_0})) \quad resp. \quad ev' : T^0(\xi, (\mathcal{O}_S, J_\sigma)) \to T^0(\xi, (\mathbb{C}, \mathfrak{m}_{X_0}))$$

is surjective.

(2) If $M \subseteq S$ is modular, write simply $\hat{\xi}$ (and $\hat{\sigma}$) for the restrictions of ξ (and σ) to M. Then

$$ev'_{|M} : T^0(\hat{\xi}) \to T^0(\hat{\xi}, (\mathbb{C}, \mathcal{O}_{X_0})) \quad resp. \quad ev'_{|M} : T^0(\hat{\xi}, (\mathcal{O}_M, J_{\hat{\sigma}})) \to T^0(\hat{\xi}, (\mathbb{C}, \mathfrak{m}_{X_0}))$$

is surjective, too.

Proof. First suppose ξ is semi-universal, with section σ. Take a derivation $\delta \in T^0(\xi, (\mathbb{C}, \mathfrak{m}_{X_0}))$. Since ξ is semi-universal, in particular we have that θ^s_ξ is injective, so we may interpret δ as element of $T^0(X_0, \mathfrak{m}_{X_0})$ and it suffices to find a preimage in $T^0(\xi, (\mathcal{O}_S, J_\sigma))$.

We use the construction of [Pal90b, Proposition 1.8]: δ induces an automorphism a_δ of $X_0 \times D$ over D. Denote by $\tilde{\xi}$ the deformation $\xi \times id_D : X \times D \to S \times D$ with section $\sigma \times id_D$. Let $T := (\{0\} \times D) \cup (S \times \{0\})$ and $\varphi : T \to S$ be the canonical projection onto S, so $\tilde{\xi}_{|T} = \varphi^*(\xi)$. Let $\tilde{a}_\delta$ be the automorphism of $\varphi^*(\xi)$ induced by a_δ on the first component of T and by id_S on the second. Then $\xi' := \tilde{a}_\delta \circ \varphi^*(\xi)$ is of course still a versal deformation with section (isomorphic to $\tilde{\xi}_{|T}$), hence $\tilde{\xi}$ is induced from it by some $\psi : S \times D \to X$, which altogether yields the diagram

$$\begin{array}{ccccc} a' : X \times D & \longrightarrow & (X \times D)_{|T} & \xrightarrow{pr_X \circ \tilde{a}_\delta} & X \\ \downarrow \tilde{\xi} & & \downarrow \xi' & & \downarrow \xi \\ \psi' : S \times D & \xrightarrow{\psi} & T & \xrightarrow{\varphi} & S. \end{array}$$

The corresponding map $\psi'^* : \mathcal{O}_S \to \mathcal{O}_S[\varepsilon]$ gives a vector field $\eta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_S)$ defined by $\psi'^*(a) = a + \varepsilon\eta(a)$, similarly a' induces a vector field $\tilde{\delta} \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X, J_\sigma)$, with the property that $(\eta, \tilde{\delta}) \in T^0(\xi, (\mathcal{O}_S, J_\sigma))$ and $ev'(\eta, \tilde{\delta}) = \delta$.

Now let $\hat{\xi} : Y \to M$ be a modular deformation with section $\hat{\sigma}$, $\hat{\xi} = \xi_{|M}$ being the restriction to $M \subseteq S$ of a semi-universal deformation with section $\xi : X \to S$ of X_0. As before, let $\tilde{\xi} := \hat{\xi} \times id_D$, and for $\delta \in T^0(\hat{\xi}, (\mathbb{C}, \mathfrak{m}_{X_0})) \simeq T^0(X_0, \mathfrak{m}_{X_0})$ let ξ' be the (still versal) deformation defined in the proof of (1). So we can induce $\tilde{\xi} : Y \times D \to M \times D$ from ξ' by means of some morphism ψ, and we obtain again:

$$\begin{array}{ccccc} a' : Y \times D & \longrightarrow & (X \times D)_{|T} & \xrightarrow{pr_X \circ \tilde{a}_\delta} & X \\ \downarrow \tilde{\xi} & & \downarrow \xi' & & \downarrow \xi \\ \psi' : M \times D & \xrightarrow{\psi} & T & \xrightarrow{\varphi} & S. \end{array}$$

On the other hand we can induce $\tilde{\xi}$ from ξ by just taking the projection $pr_1 : M \times D \to M \subseteq S$. Thus, by the modularity of M, $pr_1 = \psi'$, so that a' and ψ' can be considered as maps $Y \times D \to Y$ resp. $M \times D \to M$, inducing the desired lift of δ to $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_Y, J_{\hat{\sigma}})$ resp. $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_M)$.

Finally, omitting any section occurring we obtain the analogous statement for the functor Def_{X_0}. ∎

Putting the last assertions together, we are now able to prove the following criteria for $M \subseteq S$ to be modular:

Theorem 2.2.4. *For a subspace $M \subseteq S$ of the base space of a semi-universal deformation $\xi : X \to S$ of X_0, the following assertions are equivalent:*

(1) M is modular.

(2) M is infinitesimally modular.

(3) $ev_{|M} : T^0(X/S)_{|M} \to T^0(X_0)$ is surjective.

(4) M is Artinian modular.

The implications (3)⇒(4)⇒(1)⇒(2) remain true for any (not necessarily versal) monodeformation.

Theorem 2.2.5. *For a subspace $M \subseteq S$ of the base space of a semi-universal deformation $\xi : X \to S$ of X_0 with section $\sigma : S \to X$, the following assertions are equivalent:*

(1) M is modular.

(2) M is infinitesimally modular.

(3) $ev_{|M} : T^0(X/S, J_\sigma)_{|M} \to T^0(X_0, \mathfrak{m}_{X_0})$ is surjective.

(4) M is Artinian modular.

The implications (3)⇒(4)⇒(1)⇒(2) remain true for any (not necessarily versal) monodeformation.

A statement for deformations of compact complex spaces analogous to assertion (3) is already contained in [Pal78]. In [KS90] modular subspaces for general deformation groupoids are studied, and an analogue for this criterion is given, involving a so-called *exponential functor* that generalizes the module of derivations $T^0(X/S)$ and its interplay with the group of automorphisms of the deformation.

Similar statements have been formulated in [Mar02a], [Mar03], and a first version of the proof below was included in [HM03]. In particular, we link two criteria for modularity:

- Any vector field of the special fibre lifts to a relative vector field over M (surjectivity of $ev_{|M}$), this goes back to Palamodov's approach of proving the existence of maximal modular subspaces for deformations of compact complex spaces in [Pal78].
- The kernel of the Kodaira-Spencer map $\tilde{\Theta}_\xi$ vanishes. This is a characterization of the *prorepresenting substratum* in [LP88, Proposition 3.12], where this concept (which is an analogue within the framework of (formal) deformations of affine schemes of what we call the modular stratum here) is studied extensively by Laudal and Pfister.

We only give a proof of Theorem 2.2.5 below – Theorem 2.2.4 follows by omitting any section occurring and replacing any $T^i(X_0, \mathfrak{m}_{X_0})$ by $T^i(X_0)$, $T^i(X/S, J_\sigma)$ by $T^i(X/S)$ etc.

Proof (of Theorem 2.2.5). (1) $\Rightarrow$ (2) : This is Lemma 2.2.2 (1).

(2) $\Rightarrow$ (3) : Consider the Kodaira-Spencer sequence (2.1.2) of (ξ, σ) with coefficients in $(\mathcal{O}_M, J_{\sigma|M})$. This sequence can be put between the rows of the original sequence (2.1.2) so that altogether we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^0(X/S, J_\sigma) & \longrightarrow & T^0(\xi, (\mathcal{O}_S, J_\sigma)) & \longrightarrow & T^0(S) & \xrightarrow{\Theta^s_\xi} & T^1(X/S, J_\sigma) \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle \widetilde{ev}} & & \downarrow{\scriptstyle \widetilde{ev}'} & & \downarrow & & \downarrow \\
0 & \longrightarrow & T^0(X/S, J_\sigma)_{|M} & \longrightarrow & T^0(\xi, (\mathcal{O}_M, J_{\sigma|M})) & \longrightarrow & T^0(S, \mathcal{O}_M) & \xrightarrow{\tilde{\Theta}^s_{\xi|M}} & T^1(X/S, J_\sigma)_{|M} \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle ev_{|M}} & & \downarrow{\scriptstyle ev'_M} & & \downarrow & & \downarrow \\
0 & \longrightarrow & T^0(X_0, \mathfrak{m}_{X_0}) & \longrightarrow & T^0(\xi, (\mathbb{C}, \mathfrak{m}_{X_0})) & \longrightarrow & T^0(S, \mathbb{C}) & \xrightarrow{\theta^s_\xi} & T^1(X_0, \mathfrak{m}_{X_0}) \longrightarrow \cdots
\end{array} \tag{2.2.1}$$

By assumption $\tilde{\Theta}^s_{\xi|M}$ is injective, $\theta^s_{\xi|M}$ is injective since ξ is a monodeformation. Thus in (2.2.1) we can identify $ev_{|M}$ and ev'_M, the latter being surjective since, by Lemma 2.2.3 (1), the composition $ev' = ev'_M \circ \widetilde{ev}'$ is surjective. Note that here (and only here!) we need the versality of ξ, in order to apply Lemma 2.2.3.

(3) $\Rightarrow$ (4) : By Proposition 2.1.9 it suffices to show the following: For any Artinian germ Z and any morphism $\varphi : Z \to M$, we can extend any automorphism of $\varphi^*(\xi_{|M})_{|Z_0}$ to an automorphism of $\varphi^*(\xi_{|M})$, where $Z_0 \subseteq Z$ is a small extension. Denote $\varphi^*(\xi_{|M})$ by $\hat{\xi} : \hat{X} \to Z$, with section $\hat{\sigma}$. We explain how this follows from the assumption: We claim that (3) implies that the morphism in cotangent cohomology $T^0(\hat{X}/Z, J_{\hat{\sigma}}) \to T^0(\hat{X}/Z, J_{\hat{\sigma}})_{|Z_0}$ is surjective as well. If this holds, we are done: Any given automorphism of $\varphi^*(\xi_{|M})_{|Z_0}$ is induced by an element of $T^0(\hat{X}/Z, J_{\hat{\sigma}})_{|Z_0}$, and by the claim we can lift it to Z, so we obtain a lift to an automorphism of $\varphi^*(\xi_{|M})$. Then we can apply Proposition 2.1.9 and it follows that M is modular.

Thus it remains to give an argument for the claim. We follow the proof given in [Pal78, section 6]. Let $N \subseteq M$ be an Artinian germ such that we can factor φ as $Z \to N \hookrightarrow M$. We denote $Z \to N$ by the same letter φ. In addition, let $\pi : Z \times N \to N$ denote the projection and $G \subseteq Z \times N$ the graph of φ. We may assume that the extension $Z_0 \subseteq Z$ is simple. In particular we now have an induced deformation $\pi^*(\xi_{|N}) : Y \to Z \times N$ of X_0, together with a section $\tilde{\sigma} := \pi^*(\sigma)$. It is not hard to see that the morphisms in cotangent cohomology $T^0(Y/(Z \times N), J_{\tilde{\sigma}}) \to T^0(Y/(Z \times N), J_{\tilde{\sigma}})_{|G}$ and $T^0(Y/(Z \times N), J_{\tilde{\sigma}})_{|Z_0 \times N} \to T^0(Y/(Z \times N), J_{\tilde{\sigma}})_{|G \times_Z Z_0}$ are both surjective. These morphisms can be combined into a commutative diagram

$$\begin{array}{ccc}
T^0(Y/(Z \times N), J_{\tilde{\sigma}}) & \longrightarrow & T^0(Y/(Z \times N), J_{\tilde{\sigma}})_{|Z_0 \times N} \\
\downarrow & & \downarrow \\
T^0(Y/(Z \times N), J_{\tilde{\sigma}})_{|G} & \longrightarrow & T^0(Y/(Z \times N), J_{\tilde{\sigma}})_{|G \times_Z Z_0}
\end{array}$$

whose upper horizontal arrow is surjective as well as $Z_0 \subseteq Z$ is a simple extension. Thus we conclude the surjectivity of the lower horizontal arrow. Now consider the commutative

diagram

$$\begin{array}{ccc} G \times_Z Z_0 & \hookrightarrow & G \\ \downarrow & & \downarrow \\ Z_0 & \hookrightarrow & Z \end{array}$$

whose vertical arrows are restrictions of π. Since they induce isomorphisms of the deformations $\hat{\xi}_{|Z_0} \simeq \varphi^*(\xi_{|N})_{|Z_0} \simeq \pi^*(\xi_{|N})_{|G\times_Z Z_0}$ and $\hat{\xi} \simeq \varphi^*(\xi_{|N}) \simeq \pi^*(\xi_{|N})_{|G}$, respectively, we are done.

(4) $\Rightarrow$ (1) : This is proved in Lemma 2.2.2 (2). ∎

Remark 2.2.6.

(1) The definition of infinitesimal modularity used here differs slightly from the one stated in [Mar02a], [Mar03], where it is formulated as injectivity of the relative Kodaira-Spencer map $\Theta_{\xi|M} : T^0(M) \to T^1(X/S)_{|M}$ of the deformation $\xi_{|M}$. For any subspace $M \subseteq S$ there is a commutative diagram

$$\begin{array}{ccc} T^0(S, \mathcal{O}_M) & \xrightarrow{\tilde{\Theta}_{\xi|M}} & T^1(X_{|M}/M), \\ \uparrow & \nearrow_{\Theta_{\xi|M}} & \\ T^0(M) & & \end{array} \tag{2.2.2}$$

so $Ker(\tilde{\Theta}_{\xi|M}) = 0$ is the stronger condition, and it is indeed this property that we have to use in the above proof of the implication (2) $\Rightarrow$ (3).

(2) We can prove implication (2) $\Rightarrow$ (3) of the above statements only for semi-universal ξ (this agrees with the results in [Pal78]; in [LP88] attention is restricted exclusively to semi-universal deformations in that context). So what we can only say is that for a non-versal monodeformation $\xi : X \to S$ we still have that

- injectivity of $\tilde{\Theta}_{\xi|M}$ (resp. $\tilde{\Theta}^s_{\xi|M}$) is necessary for $M \subseteq S$ to be modular,
- surjectivity of ev (i. e. we can lift all vector fields of the special fibre to relative vector fields of the deformation $\xi_{|M}$) is sufficient for $M \subseteq S$ to be modular.

This makes it difficult to compare modular strata for the deformation functors Def_{X_0} and $Def^s_{X_0}$ – we will come back to that issue at the end of the chapter.

Criterion (2) of Theorem 2.2.4 resp. Theorem 2.2.5 gives rise to the following theoretical construction of the modular stratum of an isolated singularity: Let $\mathcal{O}_S = \mathbb{C}\{s_1, \ldots, s_\tau\}/I$ be the analytic algebra of the base space of a semi-universal deformation of X_0 and let

$$\left\{ g_{i1} \tfrac{\partial}{\partial s_1} + \ldots + g_{i\tau} \tfrac{\partial}{\partial s_\tau} \right\}_{i=1\ldots r}$$

be a set of generators of the (finitely generated) $\mathcal{O}_S$-module $Ker(\Theta_\xi) \subseteq T^0(S)$. Then $I_M := (g_{ij} : \ 1 \leq i \leq r, \ 1 \leq j \leq \tau) \bmod I$ defines a minimal ideal in $\mathcal{O}_S$ such that $\tilde{\Theta}_{\xi|M}$ is injective. Hence the subgerm $V(I_M) := M \subseteq S$ is a maximal modular subspace of S. Analogously, one takes generators of the kernel of the Kodaira-Spencer map $\Theta^s_\xi : T^0(S) \to T^1(X/S, J_\sigma)$ in the case of a semi-universal deformation with section σ. Thus we have proved:

Corollary 2.2.7. *For every isolated singularity X_0 there exists the modular stratum with respect to both deformation functors Def_{X_0} and $Def^s_{X_0}$.*

On the other hand, the above construction is only a theoretical one since, in general, there is no algorithm to compute the kernel of the relative Kodaira-Spencer map of a semi-universal deformation (the algorithms presented in [LP88] and [FK00] only compute the Kodaira-Spencer map of restricted deformations, fixing some invariants of the original singularity). Thus our construction to compute the modular stratum in chapter three will not use this characterization but merely rely on some consequences of criterion (3) that will be discussed later on in section 2.5.

2.2.3 Lifting obstructions

As a preparation we are now going to introduce an 'obstruction calculus' for the lifting of modularity under small extensions of Artinian germs, i. e. we rewrite condition (3) of the above theorems in terms of defining equations. This can be seen as an analogue of the obstruction calculus for killing obstructions in the construction of versal deformations of singularities, cf. Remark 1.2.10. In this case the obstructions are induced by the Lie bracket $T^1(X_0) \times T^1(X_0) \to T^2(X_0)$, whereas we now consider the bracket in degree one less, namely $T^0(X_0) \times T^1(X_0) \to T^1(X_0)$.

So let be given a deformation $\xi : X \to S$ of the isolated singularity X_0, and let

$$\eta : \ 0 \to J \to \mathcal{O}_B \to \mathcal{O}_A \to 0$$

be a small extension of Artinian germs $A \subseteq B \subseteq S$, i. e. $\mathfrak{m}_B J = 0$. Let $m_1, \ldots, m_t$ form a $\mathbb{C}$-basis of J. If X_0 is defined by $f \in \mathbb{C}\{x\}^k$ and X is given by $F \in \mathcal{O}_S\{x\}^k$, then we may assume that for the restrictions F_B resp. F_A of F to $\mathcal{O}_B\{x\}^k$ resp. $\mathcal{O}_A\{x\}^k$ we have

$$F_B = F_A + \sum_{j=1}^{t} m_j \alpha_j,$$

where $\alpha_j \in \mathbb{C}\{x\}^k$ represent elements of $T^1(X_0)$ (cf. [dJP00, section 10.2]). Now, for $\delta \in T^0(X_0)$, the Lie bracket

$$[-,-] : T^0(X_0) \times T^1(X_0) \to T^1(X_0)$$

gives an element $\sum_j m_j[\delta, \alpha_j] \in T^1(X_0) \otimes J$, cf. [Pal78, sections 3–4]. Altogether we obtain an induced map

$$ob_{\xi,\eta} \in \mathrm{Hom}(T^0(X_0), T^1(X_0) \otimes J) = \mathrm{Hom}(T^0(X_0), T^1(X_0)) \otimes J.$$

Definition 2.2.8. Let $\xi : X \to S$ be a deformation of X_0, and

$$\eta : 0 \to J \to \mathcal{O}_B \to \mathcal{O}_A \to 0$$

a small extension of Artinian germs $A \subseteq B \subseteq S$. We call

$$ob_{\xi,\eta} \in \mathrm{Hom}(T^0(X_0), T^1(X_0)) \otimes J$$

as defined above the *obstruction element* associated to η (with respect to the deformation ξ).

If, in addition, a section σ of ξ is given, we define similarly an obstruction element

$$ob^s_{\xi,\eta} \in \mathrm{Hom}(T^0(X_0, \mathfrak{m}_{X_0}), T^1(X_0, \mathfrak{m}_{X_0})) \otimes J.$$

The well-definedness of $ob_{\xi,\eta}$ and $ob^s_{\xi,\eta}$ follows from the fact that the Lie bracket in cotangent cohomology is independent from the choices made above. The link between this obstruction element and modular subspaces is established in the following statement:

Proposition 2.2.9. *Let $\xi : X \to S$ be a deformation of X_0 (with section $\sigma : S \to X$). Suppose that $A \subseteq B$ is a small extension of Artinian subgerms of S by means of some small extension η. If $ev_{|A} : T^0(X/S)_{|A} \to T^0(X_0)$ (resp. $T^0(X/S, J_\sigma)_{|A} \to T^0(X_0, \mathfrak{m}_{X_0})$ is surjective, then $ev_{|B} : T^0(X/S)_{|B} \to T^0(X_0)$ (resp. $ev_{|B} : T^0(X/S, J_\sigma)_{|B} \to T^0(X_0, \mathfrak{m}_{X_0})$) is surjective if and only if the obstruction element $ob_{\xi,\eta}$ (resp. $ob^s_{\xi,\eta}$) vanishes.*

Proof. We give a proof for the case of deformations with section. We may assume that the small extension is simple, i. e. given by a short exact sequence

$$\eta : 0 \to J \to \mathcal{O}_B \to \mathcal{O}_A \to 0,$$

such that $\dim_{\mathbb{C}} J = 1$, and we write ε for a generator of J. As before, let $f \in \mathcal{O}_n^k$ define X_0 and $F \in (\mathcal{O}_n \otimes \mathcal{O}_S)^k$ define X. The restrictions can be written as $F_{|B} = F_{|A} + \varepsilon\tilde{f}$ with $\varepsilon\tilde{f} \in \mathcal{O}_n^k \otimes J$ defining a class in $T^1(X_0, \mathfrak{m}_{X_0}) \otimes J$. Any $\delta \in T^0(X_0, \mathfrak{m}_{X_0})$ is represented by some $\delta' \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_n, (x))$ with $\delta'(f) = h \cdot f$, where h is a $k \times k$-matrix with entries in $\mathcal{O}_n$. By assumption we can lift δ to a derivation $\delta_A \in T^0(X/S, J_\sigma)_{|A}$, represented by some δ'_A such that $\delta'_A(F_{|A}) = H_A F_{|A}$, H_A being a lift of h to $\mathcal{O}_n \otimes \mathcal{O}_A$. Writing the same equation in $\mathcal{O}_n \otimes \mathcal{O}_B$ implies

$$\delta'_B(F_{|B}) - H_B F_{|B} = (\delta'(\tilde{f}) - h\tilde{f})\varepsilon = ob^s_{\xi,\eta}(\delta) \in T^1(X_0, \mathfrak{m}_{X_0}) \otimes J,$$

where we take *arbitrary* lifts δ'_B and H_B of δ'_A and H_A over $\mathcal{O}_B$. Thus, δ extends further to a derivation $\delta_B \in T^0(X/S, J_\sigma)_{|B}$ if and only if $ob^s_{\xi,\eta}(\delta) = 0$. ∎

As a corollary we obtain immediately:

Corollary 2.2.10. *The tangent space $T(M)$ to the modular stratum M of X_0 equals the subspace*

$$\{t \in T(S) : \ [\theta_\xi(t), \delta] = 0 \text{ for all } \delta \in T^0(X_0)\}$$

(resp. $T^0(X_0, \mathfrak{m})$ for deformations with section).

Here we use the isomorphism $T(S) \simeq T^0(S, \mathbb{C}) \simeq T^1(X_0)$ (resp. $T^1(X_0, \mathfrak{m}_{X_0})$) induced by the Kodaira-Spencer map of a semi-universal deformation.

Suppose A is modular and $\eta : 0 \to J \to \mathcal{O}_B \to \mathcal{O}_A \to 0$ is a small extension of Artinian subgerms of S, the base space of a semi-universal deformation of X_0. In addition, take generators $\delta_1, \ldots, \delta_m$ of $T^0(X_0)$ and a $\mathbb{C}$-basis $b_1, \ldots, b_\tau$ of $T^1(X_0)$. Write $F_B = F_A + \sum_j m_j g_j$, where the m_j form a $\mathbb{C}$-basis of J and g_j induce classes in $T^1(X_0)$. Then we get, for $i = 1, \ldots, m$,

$$ob_{\xi,\eta}(\delta_i) = \sum_j m_j[\delta_i, g_j] = \sum_j m_j \sum_{k=1}^{\tau} c_{ijk} b_k = \sum_{k=1}^{\tau} \left(\sum_j c_{ijk} m_j \right) b_k = \sum_{k=1}^{\tau} h_{ik} b_k,$$

for suitable $c_{ijk} \in \mathbb{C}$ and $\sum_j c_{ijk} m_j =: h_{ik} \in J$. Define J' to be the ideal generated by all h_{ik} in $\mathcal{O}_B$. Then $ob_{\xi,\eta}$ vanishes modulo J', so the subgerm B' of S with local ring $\mathcal{O}_B/J'$ is modular, lies between A and B, and is clearly maximal with this property. Altogether we get:

Corollary 2.2.11. *With the notations introduced before:*

(1) The subgerm $B' \subseteq S$ with local algebra $\mathcal{O}_{B'} = \mathcal{O}_B/J'$ is modular and

(2) B' is maximal among the subgerms of S lying between A and B and satisfying condition (1).

The same statement is true when considering a semi-universal deformation with section and $T^i(X_0, \mathfrak{m}_{X_0})$, $ob^s_{\xi,\eta}$ instead.

Remark 2.2.12. Note the apparent similarity of this construction to the computation of semi-universal deformations of singularities, where an analogous consideration gives the maximal extension of a deformation of order k to a deformation of order $k+1$, cf. [dJP00, section 10.3], for example. In this case, the obstruction space is given as $T^2(X_0) \otimes J$, whereas in our context the obstructions belong to $T^1(X_0) \otimes J$.

2.3 The support of the modular stratum

We now turn our attention to the underlying space forming the modular stratum $M \subseteq S$, i. e. we pass to small representatives of S, M and X. The result will be that, for a suitably chosen representative of S, $M \subseteq S$ consists precisely of those points $s \in S$ such that the fibre X_s has still the same Tjurina number as the original singularity X_0. For compact complex spaces, it is also known that this subspace of S coincides with the stratum of points where the dimension of $T^0(X_0)$ is constant by [Pal78, Theorem 7.2]. In general, for isolated singularities, $T^0(X_0)$ is not finite-dimensional, so this statement does no longer make sense in the local situation. But by considering a suitable quotient of $T^0(X_0)$ instead we will end up with a module which is finitely generated as a $\mathbb{C}$-vector space, and for which similar properties hold.

2.3.1 The modules $T^0_\bullet(X_0)$ and $T^0_\bullet(X_0, \mathfrak{m}_{X_0})$

So let us start by introducing this module: Obviously, in the modularity criteria Theorem 2.2.4 and Theorem 2.2.5, surjectivity of the evaluation mappings is equivalent to the surjectivity of the mappings

- $T^0(X/S)_{|M} \to T^0(X_0)/ev(T^0(X/S))$ resp.
- $T^0(X/S, J_\sigma)_{|M} \to T^0(X_0, \mathfrak{m})/ev(T^0(X/S, J_\sigma))$.

This gives rise to the following notions (cf. [Pal90b]):

Definition 2.3.1. Let $\xi : X \to S$ be a monodeformation of the singularity X_0 (with section σ). We define $T^0_\xi(X_0)$ resp. $T^0_\xi(X_0, \mathfrak{m}_{X_0})$ as the quotient

$$Im(ev')/Im(ev)$$

in the diagrams (2.1.1) resp. (2.1.2). This makes sense because $T^0(X_0) \simeq T^0(\xi, (\mathbb{C}, \mathcal{O}_{X_0}))$ resp. $T^0(X_0, \mathfrak{m}_{X_0}) \simeq T^0(\xi, (\mathbb{C}, \mathfrak{m}_{X_0}))$ for a monodeformation, i. e. we have the diagrams

$$\begin{array}{ccc} T^0(X/S) & \hookrightarrow & T^0(\xi) \\ \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev'} \\ T^0(X_0) & \xrightarrow{\sim} & T^0(\xi, (\mathbb{C}, \mathcal{O}_{X_0})) \end{array} \qquad \text{resp.} \qquad \begin{array}{ccc} T^0(X/S, J_\sigma) & \hookrightarrow & T^0(\xi, (\mathcal{O}_S, J_\sigma)) \\ \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev'} \\ T^0(X_0, \mathfrak{m}_{X_0}) & \xrightarrow{\sim} & T^0(\xi, (\mathbb{C}, \mathfrak{m}_{X_0})). \end{array}$$

If ξ is semi-universal, we simply write

- $T^0_\bullet(X_0) := T^0(X_0)/ev(T^0(X/S))$, resp.
- $T^0_\bullet(X_0, \mathfrak{m}_{X_0}) := T^0(X_0, \mathfrak{m}_{X_0})/ev(T^0(X/S, J_\sigma))$ for the case of deformations with section,

which equals T^0_ξ in this case because ev' is surjective for a semi-universal deformation by Proposition 2.2.3.

The following properties of $T^0_\bullet(X_0)$ go back to [Pal90b]:

Proposition 2.3.2. *Let X_0 be an isolated singularity. Then:*

(1) $\dim_{\mathbb{C}} T^0_\xi(X_0) < \infty$ *for any monodeformation ξ of X_0.*

(2) If X_0 is a hypersurface defined by $f \in \mathcal{O}_n$, then there is an exact sequence of $\mathcal{O}_n$-modules

$$0 \to T^0_\bullet(X_0) \xrightarrow{\varepsilon} \mathcal{O}_n/J(f) \xrightarrow{m_f} \mathcal{O}_n/J(f) \xrightarrow{\pi} T^1(X_0) \to 0, \tag{2.3.1}$$

where π is the projection, m_f denotes multiplication by f modulo $J(f)$, and ε is defined as follows: If $\delta \in T^0(X_0)$ is the class of a derivation $\delta' : \mathcal{O}_n \to \mathcal{O}_n$ subject to $\delta'(f) = h \cdot f$ for some $h \in \mathcal{O}_n$, set $\varepsilon(\delta) := \frac{\delta'(f)}{f} = h \bmod J(f)$.

(3) More generally, if $T^2(X_0) = 0$ and $\xi : X \to S$ is a semi-universal deformation of X_0, then, for all points s in a small representative of S:

$$\dim_{\mathbb{C}} Ker(\Theta_\xi)_0 \otimes_{\mathcal{O}_{S,0}} \mathcal{O}_{S,0}/\mathfrak{m}_{S,0} - \dim_{\mathbb{C}} Ker(\Theta_\xi)_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S,s}/\mathfrak{m}_{S,s} = \left[\dim_{\mathbb{C}} T^0_\bullet(X_0) - \dim T^1(X_0)\right] - \left[\dim_{\mathbb{C}} T^0_\xi(X_s) - \dim_{\mathbb{C}} T^1(X_s)\right].$$

Here X_s denotes the multigerm of the fibre of a representative $\xi : X \to S$ over $s \in S$.

In particular, this implies the inequality

$$\dim_{\mathbb{C}} T^0_\bullet(X_0) \geq \tau(X_0),$$

with equality if and only if $Ker(\Theta_\xi)$ is a free $\mathcal{O}_S$-module.

The latter is the case if X_0 is a complete intersection or if X_0 is a reduced space curve singularity.

Proof. The assertions (1)–(3) are proven in [Pal90b], Proposition 1.7, Proposition 3.1 and Theorem 2.3. The Freeness of $Ker(\Theta_\xi)$ for complete intersections is established by the Saito-Looijenga-Theorem, see [Sai80] or [Loo84], and for reduced space curves this has been proven by van Straten in [vS95]. ∎

We can prove properties similar to (1) and (2) for the module $T^0_\bullet(X_0, \mathfrak{m})$.

Proposition 2.3.3. *Let X_0 be an isolated singularity. Then:*

(1) $\dim_{\mathbb{C}} T^0_\xi(X_0, \mathfrak{m}_{X_0}) < \infty$ *for any monodeformation with section of X_0.*

(2) If X_0 is a hypersurface defined by $f \in \mathcal{O}_n$, then there is an exact sequence of $\mathcal{O}_n$-modules

$$0 \to T^0_\bullet(X_0, \mathfrak{m}_{X_0}) \xrightarrow{\varepsilon} \mathcal{O}_n/J(f) \xrightarrow{m_f} \mathcal{O}_n/(x)J(f) \xrightarrow{\pi} \mathcal{O}_n/(f,(x)J(f)) \to 0, \qquad (2.3.2)$$

where the maps ε, m_f and π are defined analogously to Proposition 2.3.2 (2). In particular we have:

$$\dim_{\mathbb{C}} T^0_\bullet(X_0, \mathfrak{m}_{X_0}) = \dim_{\mathbb{C}} T^0_\bullet(X_0) = \tau(X_0)$$

for any isolated hypersurface singularity.

Proof. Property (1) is a consequence of the Kodaira-Spencer sequence (2.1.2), cf. the proof of [Pal90b, Proposition 1.7]: Its beginning can be written as

$$\begin{array}{ccccccc} 0 & \longrightarrow & T^0(X/S, J_\sigma) & \longrightarrow & T^0(\xi, (\mathcal{O}_S, J_\sigma) & \longrightarrow & Ker(\Theta^s_\xi) \longrightarrow 0 \\ & & \downarrow ev & & \downarrow ev' & & \\ 0 & \longrightarrow & T^0(X_0, \mathfrak{m}_{X_0}) & \xrightarrow{\sim} & T^0(\xi, (\mathbb{C}, \mathfrak{m}_{X_0})) & \longrightarrow & 0, \end{array}$$

so ev' induces a surjective $\mathcal{O}_S$-linear map

$$T^0(\xi, (\mathcal{O}_S, J_\sigma))/T^0(X/S, J_\sigma) \twoheadrightarrow T^0_\xi(X_0, \mathfrak{m}_{X_0}).$$

The $\mathcal{O}_S$-module on the left is isomorphic to the submodule $Ker(\Theta^s_\xi)$ of the finitely generated $\mathcal{O}_S$-module $T^0(S)$, hence it is itself finitely generated. Thus, applying $- \otimes_{\mathcal{O}_S} \mathbb{C}$ yields a surjective map

$$T^0(\xi, (\mathcal{O}_S, J_\sigma))/T^0(X/S, J_\sigma) \otimes_{\mathcal{O}_S} \mathbb{C} \twoheadrightarrow T^0_\xi(X_0, \mathfrak{m}_{X_0}) \otimes_{\mathcal{O}_S} \mathbb{C} \simeq T^0_\xi(X_0, \mathfrak{m}_{X_0}),$$

whose source is a finite-dimensional $\mathbb{C}$-vector space, hence $\dim_{\mathbb{C}} T^0_\xi(X_0, \mathfrak{m}_{X_0}) < \infty$, too.

We now prove (2), which can be done analogously to [Pal90b, Proposition 3.1]. First we show that ε is well-defined: Let $f \in \mathcal{O}_n$ define X_0 and let X be defined by $F \in \mathcal{O}_S\{x\}$. In addition we may assume that σ is the zero section, i. e. $J_\sigma = (x)\mathcal{O}_S\{x\}/(F)$. If $\delta \in T^0(X_0, \mathfrak{m}_{X_0})$ lifts to $\tilde{\delta} \in T^0(X/S, J_\sigma)$, then there exist $G_i \in (x)\mathcal{O}_S\{x\}$ and $H \in \mathcal{O}_S\{x\}$ such that $\tilde{\delta}$ is the class of $\sum_i G_i \frac{\partial}{\partial x_i}$ and $\sum_i G_i \frac{\partial F}{\partial x_i} = H \cdot F$. Since F and its partial derivatives with respect to the x_i form a regular sequence in $\mathcal{O}_S\{x\}$, this implies that H is in the ideal generated by $\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}$, i. e. $\varepsilon(\delta) = H_{|s=0} \in J(f)$.

It remains to verify the exactness of the sequence, which is obvious at the two terms on the right. Also, it is clear that $Im(\varepsilon) \subseteq Ker(m_f)$. On the other hand, if $h \bmod J(f) \in Ker(m_f)$, then $hf = \sum_i g_i \frac{\partial f}{\partial x_i}$ with $g_i \in (x)$, i. e. $h = \varepsilon(\delta)$ for the derivation $\delta : \mathcal{O}_{X_0} \to \mathfrak{m}_{X_0}$ induced by $\sum_i g_i \frac{\partial}{\partial x_i}$. Thus, $Ker(m_f) = Im(\varepsilon)$, and it is only left to show that ε has trivial kernel. But if δ is induced by a derivation δ' of $\mathcal{O}_n$ with $\sum_i g_i \frac{\partial f}{\partial x_i} = h \cdot f$ and $h = \sum_i h_i \frac{\partial f}{\partial x_i}$, then its class equals the class of $\sum_i (g_i - f \cdot h_i) \frac{\partial}{\partial x_i}$, so we may assume that for its lift δ' we have $\delta'(f) = 0$. Thus its coefficients are a syzygy of the partial derivatives of f which form a regular sequence in $\mathcal{O}_n$. Hence δ belongs to the submodule generated by the classes of the derivations of the form $\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}$, which clearly can be lifted to $T^0(X/S, J_\sigma)$.

The dimension of $T^0_\bullet(X_0, \mathfrak{m}_{X_0})$ is now a consequence of Proposition 1.3.4, which implies that

$$\dim_{\mathbb{C}} \mathcal{O}_n/(f,(x)J(f)) = \dim_{\mathbb{C}} T^1(X_0, \mathfrak{m}_{X_0}) + 1 = \tau(X_0) + n.$$

In addition, let $\mu(X_0) := \dim_{\mathbb{C}} \mathcal{O}_n/J(f)$ be the Milnor number of X_0. Corollary 1.3.5 states that the partial derivatives of f are linearly independent modulo $(x)J(f)$, hence the dimension of $\mathcal{O}_n/(x)J(f)$ equals $\mu + n$. Altogether we calculate using (2.3.2):

$$\begin{aligned}\dim_{\mathbb{C}} T^0_\bullet(X_0) &= \dim_{\mathbb{C}} \mathcal{O}_n/J(f) - \dim_{\mathbb{C}} \mathcal{O}_n/(x)J(f) + \dim_{\mathbb{C}} \mathcal{O}_n/(f,(x)J(f)) \\ &= \mu(X_0) - (\mu(X_0)+n) + (\tau(X_0)+n) \\ &= \tau(X_0),\end{aligned}$$

which proves the last assertion. ∎

2.3.2 The support of M_{X_0}

We finish this section by giving the characterization of the points in $M_{X_0} \subseteq S$ already proposed earlier in this chapter. Since in this context we talk about the points in representatives of X, S and M, we are more careful regarding notations and mark the base points of the occurring germs explicitly.

Proposition 2.3.4. *Let $(X_0,0)$ be an isolated singularity, $\xi : (X,0) \to (S,0)$ a semi-universal deformation and $(M,0) \subseteq (S,0)$ a maximal modular subspace. Then:*

(1) There exist representatives X and $M \subseteq S$ of these germs such that M coincides with the set of those $s \in S$ such that ξ is a semi-universal deformation of the fibre X_s.

(2) If $T^2(X_0) = 0$, then this coincides with the points $s \in M$ such that

$$\dim_{\mathbb{C}} T^1(X_s) = \dim_{\mathbb{C}} T^1(X_0).$$

In particular, for suitably chosen representatives the modular stratum of $(X_0,0)$ coincides with the stratum $\tau = const$.

(3) If, in addition, the kernel of the relative Kodaira-Spencer map $T^0(S) \to T^1(X/S)$ is free, this also equals the set of those $s \in S$ such that

$$\dim_{\mathbb{C}} T^0_\bullet(X_s) = \dim_{\mathbb{C}} T^0_\bullet(X_0).$$

Remark 2.3.5. Although we have stated (and prove) this result only for the functor Def_{X_0}, it remains true when adapted to the functor of deformations with section. For the third characterization, one needs additionally a property of $T^0_\bullet(X_0, \mathfrak{m}_{X_0})$ similar to Proposition 2.3.2 (3).

Proof. ad (1): Choose small representatives X and S of $(X,0)$ resp. $(S,0)$ such that ξ is still a versal deformation (as multigerm!) of the fibre X_s for all $s \in S$. Consider the 'sheafified' version of the right hand part of sequence (2.1.1) for arbitrary $s \in S$, as described in section 1.1.4:

$$\begin{array}{ccccc} T^0(\xi,(\mathcal{O}_M,\mathcal{O}_{|X_M}) & \longrightarrow & T^0(S,\mathcal{O}_M) & \xrightarrow{\tilde{\Theta}_{\xi|M}} & T^1(X_{|M}/M) \\ \downarrow & & \downarrow & & \downarrow \\ T^0(\xi,(\mathbb{C}_s,\mathcal{O}_{X_s})) & \longrightarrow & T^0(S,\mathbb{C}_s) & \xrightarrow{\theta_\xi} & T^1(X_s) \end{array}$$

The rows of this sequence are exact, and taking stalks at a point of S yields the corresponding sequence of germs. – In particular, its stalk at $0 \in S$ is the restriction of (2.1.1) to M. By the choice of X and S, the map $T^0(S, \mathbb{C}_s)_s \to T^1(X_s)_s$ is still surjective for any $s \in S$ and we have to show that it is injective if and only if $s \in M$.

So let $s \in M$ and suppose $\delta \in T^0(S, \mathbb{C}_s)_s$ induces a trivial infinitesimal deformation of the multigerm X_s. From the exactness of the lower row it follows that we can find a corresponding pair $(\delta, \tilde{\delta}) \in T^0(\xi, (\mathbb{C}_s, \mathcal{O}_{X_s}))_s$, which can be lifted to an element $(\eta, \tilde{\eta})$ of $T^0(\xi, (\mathcal{O}_M, \mathcal{O}_{X_{|M}}))_s$. But now η induces an element of the kernel of $(T^0(S, \mathcal{O}_M) \to T^1(X_{|M}/M))_s$, hence it must be zero by the properties of M (characterization (2) in Theorem 2.2.4) and by the fact that $Ker(\Theta_\xi)$ is coherent, therefore $\delta = 0$, too. On the other hand, if $s \in S \setminus M$ then it is clear that the existence of non-trivial elements in the kernel of the relative Kodaira-Spencer map implies that $T^0(S, \mathbb{C}_s)_s \to T^1(X_s)_s$ is not injective.
ad (2): This is now a direct consequence of (1) since, in case $T^2(X_0) = 0$, S is smooth and of dimension $\tau(X_0)$.
ad (3): Under these assumptions we can find representatives such that

$$\dim_{\mathbb{C}} T^0_\bullet(X_s) - \dim_{\mathbb{C}} T^1(X_s) = \dim_{\mathbb{C}} T^0_\bullet(X_0) - \dim_{\mathbb{C}} T^1(X_0)$$

for all $s \in S$ by Proposition 2.3.2, hence we are done by (2). ∎

2.4 Example: Quasihomogeneous complete intersections

At this point, we would like to demonstrate the constructions of this chapter by means of a simple but important example: Quasihomogeneous complete intersection singularities. In general, the modular stratum of an isolated singularity X_0 carries a non-reduced structure – we will present a lot of concrete examples in the last part of this thesis, calculated using Singular. General statements concerning the structure of M_{X_0} are difficult to obtain, because if one wants to use one of the criteria above, one either needs the kernel of the relative Kodaira-Spencer map Θ_ξ or the module of derivations $T^0(X_0)$ (or $T^0_\bullet(X_0)$) as a first step, and a general description of these modules tends to be very hard. However, Aleksandrov's article [Ale85] contains a complete, explicit description of $T^0(X_0)$ for a quasihomogeneous isolated complete intersection X_0 of positive dimension. It generalizes the well-known fact, that for quasihomogeneous hypersurface singularities the module of derivations is generated by the Euler relation together with the trivial ones (coming from the Koszul relations). From this result one can easily derive the structure of M_{X_0} and obtains that it is reduced and smooth. For the functor Def_{X_0}, this is already observed in [Ale85, section 6]. Here we extend this result to deformations with section, in particular showing that both modular strata coincide for this class of singularities.

So let X_0 be a complete intersection of positive dimension, defined by quasihomogeneous polynomials $f = (f_1, \ldots, f_k) \in \mathcal{O}_n^k$ of degrees $d_1, \ldots, d_k$ with respect to some positive integer weights $w_1, \ldots, w_n$.

Proposition 2.4.1. *For such singularities, the $\mathcal{O}_{X_0}$-module of derivations* $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X_0}) \simeq T^0(X_0)$ *is generated by the Euler derivation $\delta_E = \sum_{i=1}^n w_i x_i \frac{\partial}{\partial x_i}$, together with the* Hamiltonian derivations $H(i_1, \ldots, i_{k+1})$ *for any $1 \leq i_1 < \ldots < i_{k+1} \leq n$, which are obtained by*

cofactor expansion with respect to the first line of the symbolic matrix

$$\begin{pmatrix} \frac{\partial}{\partial x_{i_1}} & \cdots & \frac{\partial}{\partial x_{i_{k+1}}} \\ \frac{\partial f_1}{\partial x_{i_1}} & \cdots & \frac{\partial f_1}{\partial x_{i_{k+1}}} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_{i_1}} & \cdots & \frac{\partial f_k}{\partial x_{i_{k+1}}} \end{pmatrix}.$$

Proof. [Ale85, section 6.1] – in fact, this had already been announced by Wahl in [Wah83]. ∎

Proposition 2.4.2. *The modular stratum with respect to the functors Def_{X_0} and $Def^s_{X_0}$ of a quasihomogeneous isolated complete intersection singularity X_0 of positive dimension is reduced and smooth.*

Its dimension equals the dimension of the eigenspace (with respect to the eigenvalue zero) of the linear operator $[\delta_E, -] : T^1(X_0) \to T^1(X_0)$ – resp. $T^1(X_0, \mathfrak{m}_{X_0}) \to T^1(X_0, \mathfrak{m}_{X_0})$ – defined by the Euler derivation δ_E.

Proof. For the functor Def_{X_0} this result has already been proved in [Ale85, section 6.2], here we give a proof for deformations with section. First of all, note that Proposition 2.4.1 describes $T^0(X_0, \mathfrak{m}_{X_0})$ as well, since $T^0(X_0) \simeq T^0(X_0, \mathfrak{m}_{X_0})$ for any isolated singularity by Proposition 1.3.4 (of course, this also follows directly from the above description of this module).

Now let $\xi : X \to S$ be a semi-universal deformation with section σ of X_0. In particular, by the results of section 1.3, $S \simeq (\mathbb{C}^{\tau^s}, 0)$ is smooth. Obviously, all $H(i_1, \ldots, i_{k+1})$ can be lifted to $T^0(X/S, J_\sigma)$, so we only have to consider δ_E. The bracket $[-,-]$ in cotangent cohomology induces an element $[\delta_E, -] \in \mathrm{End}_{\mathbb{C}}(T^1(X_0, \mathfrak{m}_{X_0}))$ that gives a decomposition $T^1(X_0, \mathfrak{m}_{X_0}) = \bigoplus_{\nu \in \mathbb{Z}} T^1(X_0, \mathfrak{m}_{X_0})_\nu$ into a direct sum of its eigenspaces $T^1(X_0, \mathfrak{m}_{X_0})_\nu$ with respect to the eigenvalue ν. Using the description of the tangent space to the modular stratum M_{X_0} in Corollary 2.2.10 we obtain $T(M_{X_0}) \simeq T^1(X_0, \mathfrak{m}_{X_0})_0$.

But now it is clear that there are no further obstructions to lifting δ_E: If $F = f + \sum_{i=1}^{\tau^s} s_i g^{(i)}$ is a semi-universal family with monomials $g^{(i)}$ as in Remark 1.3.10, then we may assume that $g^{(1)}, \ldots, g^{(r)}$ represent a basis of $T^1(X_0, \mathfrak{m}_{X_0})_0$. From what was said above, it then follows that $T(M_{X_0}) \simeq ((s_1, \ldots, s_r)/(s_1, \ldots, s_r)^2)^*$, and we can lift δ_E to the restriction of ξ to the smooth subspace $(\mathbb{C}^r \times \{0\}, 0)$ of S, i. e. $\xi_{|(\mathbb{C}^r \times \{0\}, 0)}$ is the maximal modular deformation inside ξ. ∎

Corollary 2.4.3. *For any quasihomogeneous isolated complete intersection singularity X_0 of positive dimension the modular strata with respect to both deformation functors Def_{X_0} and $Def^s_{X_0}$ coincide.*

Remark 2.4.4. Let us show how one can obtain this result differently, by analyzing the relative Kodaira-Spencer map $\Theta_\xi : T^0(S) \to T^1(X/S)$: Let $F \in \mathbb{C}\{s, x\}^k$ define a semi-universal deformation of X_0 as above, so $F = f + \sum_{i=1}^{\tau} s_i g^{(i)}$, where we can take $g^{(i)} = x^{\alpha_i} e_{k(i)}$ for all i, and these elements induce a $\mathbb{C}$-basis of $T^1(X_0)$. By $w(\alpha)$ we denote the weight of α with respect to the given weight vector $w = (w_1, \ldots, w_n)$, and $d_1, \ldots, d_k$ are the w-degrees of $f_1, \ldots, f_k$. Then

$$\sum_{i=1}^{\tau} \left(d_{k(i)} - w(\alpha_i)\right) s_i \tfrac{\partial}{\partial s_i} \in Ker(\Theta_\xi),$$

since it is mapped under Θ_ξ to the class of

$$\sum_{i=1}^{\tau} \left(d_{k(i)} - w(\alpha_i)\right) s_i g^{(i)} = \begin{pmatrix} d_1 F_1 \\ \vdots \\ d_k F_k \end{pmatrix} - \sum_{j=1}^{n} w_j x_j \cdot \begin{pmatrix} \frac{\partial F_1}{\partial x_j} \\ \vdots \\ \frac{\partial F_k}{\partial x_j} \end{pmatrix},$$

hence gives zero in $T^1(X/S)$. Therefore the ideal

$$I := (s_i : d_{k(i)} \neq w(\alpha_i)) = (s_i : g^{(i)} \notin T^1(X_0)_0),$$

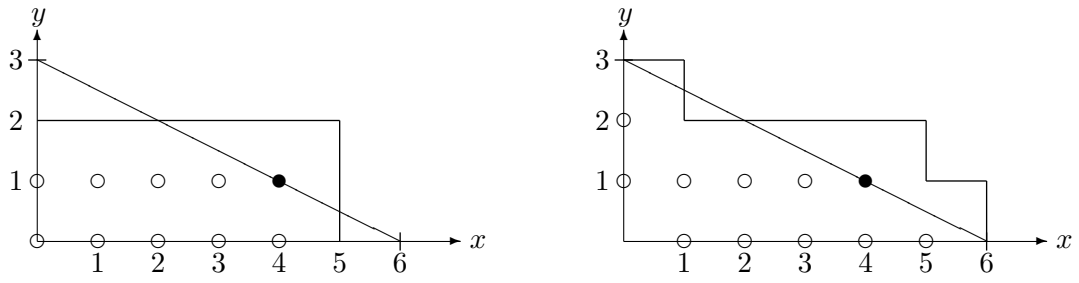
is contained in the ideal of a maximal modular subspace, where, as before, $T^1(X_0)_0$ means the eigenspace to the eigenvalue 0 of the linear operator $[\delta_E, -] \in \mathrm{End}_{\mathbb{C}}(T^1(X_0))$. Thus the subgerm $M' \subseteq S$ defined by $\mathcal{O}_{M'} := \mathcal{O}_S/I$ contains a maximal modular subspace. But M' actually is modular itself, since restricted to M' we obtain that

$$\tilde{\Theta}_{\xi|M'} : T^0(S, \mathcal{O}_S/I) \to T^1(X/S)_{|M'}$$

is injective: The latter module is equal to the quotient of $OO^k_{X_{|M'}}$ by the submodule generated by the classes of $\frac{\partial F_1}{\partial x_1}, \ldots, \frac{\partial F_k}{\partial x_j}$, $1 \leq j \leq k$. Since $F_{|M'}$ becomes quasihomogeneous, this is a free module with $\{\overline{g^{(i)}} : g^{(i)} \notin T^1(X_0)_0\}$ forming a set of free generators, and $\tilde{\Theta}_{\xi|M'}$ is given by $s_i \mapsto \overline{g^{(i)}}$. Hence M' is a maximal modular subspace of S by Theorem 2.2.5, equal to the subspace described in Proposition 2.4.2.

Of course, the same consideration is valid for deformations with section and gives a different proof of Proposition 2.4.2 for the case of the functor $Def^s_{X_0}$.

Example 2.4.5. As a simple concrete example consider the singularity defined by $f := x^6 + y^3 \in \mathbb{C}\{x, y\}$. This defines a parabolic singularity J_{10} (in Arnold's terminology, [AGZV85]), which is unimodal. Bases of the vector spaces $T^1(X_0) \simeq \mathbb{C}\{x, y\}/(x^5, y^2)$ and $T^1(X_0, \mathfrak{m}_{X_0}) \simeq (x, y)/(x^6, xy^2, xy^5, y^3)$ can be indicated as follows:



f is quasihomogeneous of degree 1 with respect to the weight vector $w = (\frac{1}{6}, \frac{1}{3})$, and x^4y is the unique monomial of weighted degree 1 in the above basis. Hence, $f + t \cdot x^4y$ defines a maximal modular deformation of f over the one-dimensional base with local ring $\mathbb{C}\{t\}$.

Remark 2.4.6. We cannot drop parts of the assumptions made on X_0. For instance, all these statements become wrong when considering zero-dimensional singularities, and we will give several examples of the phenomena that can occur in section 4.3.

2.5 Flatness conditions

Let $\xi : X \to S$ be a semi-universal deformation of the isolated singularity X_0. In section 2.2, the modular stratum $M_{X_0} =: M \subseteq S$ was characterized by the properties that all vector fields of the special fibre (the elements of $T^0(X_0)$ can be lifted to relative vector fields of the family $X_{|M} \to M$ (the elements of $T^0(X/S)_{|M}$). In terms of equations $f_1 = \ldots = f_k = 0$ defining X_0, an element $\delta \in T^0(X_0)$ is given by $g_1, \ldots, g_n \in \mathcal{O}_n$ and a $k \times k$-matrix $h = (h_{ij})$ with entries in $\mathcal{O}_n$ such that δ is the class of $\tilde{\delta} = \sum_i g_i \frac{\partial}{\partial x_i} \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_n)$ and

$$\begin{pmatrix} \tilde{\delta}(f_1) \\ \vdots \\ \tilde{\delta}(f_k) \end{pmatrix} = \begin{pmatrix} h_{11} & \ldots & h_{1k} \\ \vdots & & \vdots \\ h_{k1} & \ldots & h_{kk} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix},$$

i. e. $(h_{11}, \ldots, h_{k1}, h_{12}, \ldots, h_{kk}, -g_1, \ldots, -g_n)$ is a syzygy of the columns of the matrix

$$P_f := \begin{pmatrix} f_1 & & f_2 & & \cdots & f_k & & \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \ddots & & \ddots & \cdots & & \ddots & \vdots & & \vdots \\ & f_1 & & f_2 & \cdots & & f_k & \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix},$$

so every δ gives rise to a syzygy of the column vectors of this matrix, and M is modular if and only if every such syzygy has a lift over M. Since, over local rings, lifting of syzygies can be interpreted as flatness of the corresponding module, this gives rise to a characterization of M as flattening stratum of a suitable module. In the present section, we will make these assertions precise and carry out the necessary proofs, generalizing and, at some points, modifying previous results in this direction contained in [Mar02a], [Mar03].

2.5.1 Flatness of the relative T^1

Definition 2.5.1. Let X_0 be a singularity, minimally embedded in $(\mathbb{C}^n, 0)$ and defined by $f = (f_1, \ldots, f_k) \in \mathcal{O}_n$. Let, in addition, $\xi : X \to S$ be a deformation of X, defined by $F = (F_1, \ldots, F_k)$. We use the notations

$$J(f) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix}, \qquad J_x(F) := \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_n} \end{pmatrix}$$

for the Jacobian matrix of $f_1, \ldots, f_k$ and for the relative Jacobian matrix of $F_1, \ldots, F_k$, respectively.

Similar to the matrix P_f defined above, we denote by P_F the $(k^2 + n) \times n$-matrix

$$P_F := \begin{pmatrix} F_1 & & F_2 & & \cdots & F_k & & \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ & \ddots & & \ddots & \cdots & & \ddots & \vdots & & \vdots \\ & F_1 & & F_2 & \cdots & & F_k & \frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_n} \end{pmatrix}.$$

Now we set $\tilde{T}^1(X/S) := \mathcal{O}_X^k / J_x(F) = \mathcal{O}_S\{x\}^k / P_F$. If ξ is a deformation of X_0 with section σ, we denote by $\tilde{T}^1(X/S, J_\sigma)$ the quotient of $\mathcal{O}_X^k$ by $J_\sigma \cdot J_x(F)$, where, as usually, $J_\sigma := Ker(\sigma^*)$.

Using these notions we can interpret criterion (3) in Theorem 2.2.4 as flatness of the module $\tilde{T}^1(X/S)$, as indicated in the introduction to this section:

Proposition 2.5.2. *Let $M \subseteq S$ be a subspace in the base space of a semi-universal deformation $\xi : X \to S$ of the isolated singularity X_0. Then M is modular if and only of $\tilde{T}^1(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_M$ is a flat $\mathcal{O}_M$-module.*

Proof. There is the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \mathcal{O}_M\{x\}^{k^2+n} & \xrightarrow{P_F} & \mathcal{O}_M\{x\}^k & \longrightarrow & \tilde{T}^1(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_0 & \longrightarrow & \mathcal{O}_n^{k^2+n} & \xrightarrow{P_f} & \mathcal{O}_n^k & \longrightarrow & \mathcal{O}_{X_0}^k/J(f) \longrightarrow 0, \end{array}$$

where K and K_0 denote the kernels of the corresponding maps. By the lifting criterion for flatness (see [Mat86, § 7], for example), $\tilde{T}^1(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_M = \tilde{T}^1(X_{|M}/M)$ is a flat $\mathcal{O}_M$-module if and only if $K \to K_0$ is a surjection. We show that this is equivalent to $ev : T^0(X/S)_{|M} \to T^0(X_0)$ being surjective, which concludes the proof of the assertion using Theorem 2.2.4. Any element $\binom{h_{il}}{g_j} \in K_0$ corresponds to a derivation $\delta = \sum_j g_j \frac{\partial}{\partial x_j}$ such that $\sum_j g_j \frac{\partial f_l}{\partial x_j} + \sum_i h_{il} f_i = 0$ in $\mathcal{O}_n$ for all l, so this gives rise to a surjection $\alpha_0 : K_0 \to T^0(X_0)$, similarly we construct a surjection $\alpha : K \to T^0(X/S)_{|M}$. It is easy to see that the map

$$\beta_0 : \mathcal{O}_n^{kn} \to Ker(\alpha_0),\ (m_{ij}) \mapsto \begin{pmatrix} \left(-\sum_j m_{ij} \frac{\partial f_l}{\partial x_j}\right)_{i,l} \\ \left(\sum_i m_{ij} f_i\right)_j \end{pmatrix}$$

is again a surjection, analogously we obtain a surjection $\beta : \mathcal{O}_M\{x\}^{kn} \to K$, and we end up with the commutative diagram with exact rows

$$\begin{array}{ccccccccc} Ker(\beta) & \longrightarrow & \mathcal{O}_M\{x\}^{kn} & \xrightarrow{\beta} & K & \xrightarrow{\alpha} & T^0(X/S)_{|M} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ Ker(\beta_0) & \longrightarrow & \mathcal{O}_n^{kn} & \xrightarrow{\beta_0} & K_0 & \xrightarrow{\alpha_0} & T^0(X_0) & \longrightarrow & 0. \end{array}$$

If $K \twoheadrightarrow K_0$, then the square on the right implies that $T^0(X/S)_{|M} \twoheadrightarrow T^0(X_0)$ as well. Conversely, if $T^0(X/S)_{|M} \twoheadrightarrow T^0(X_0)$ then the surjectivity of $K \to K_0$ follows by applying the Five Lemma. ∎

If X_0 is a complete intersection then $\tilde{T}^1(X/S)$ equals $T^1(X/S)$, hence:

Corollary 2.5.3. *The modular stratum of an isolated complete intersection singularity X_0 equals the flattening stratum of the relative Tjurina module $T^1(X/S)$, where $\xi : X \to S$ is a semi-universal deformation of X_0.*

More generally, there is always an exact sequence

$$0 \to T^1(X/S) \to \tilde{T}^1(X/S) \to \mathcal{O}_X^k/\mathrm{Hom}_{\mathcal{O}_X}((F)/(F)^2, \mathcal{O}_X) \to 0,$$

which implies:

Corollary 2.5.4. *Let $\xi : X \to S$ be a semi-universal deformation of X_0 such that*

$$\mathcal{O}_X^k/\mathrm{Hom}_{\mathcal{O}_X}((F)/(F)^2, \mathcal{O}_X)$$

is a flat $\mathcal{O}_S$-module, $F \in \mathcal{O}_S\{x\}^k$ defining X. Then the modular stratum of X_0 coincides with the flattening stratum of $T^1(X/S)$.

Aside from complete intersections where the condition in the corollary is trivially satisfied (the quotient is zero), Cohen-Macaulay singularities of codimension 2 yield a further class of examples. This follows from the determinantal structure of the normal module $N_X := \mathrm{Hom}_{\mathcal{O}_X}((F)/(F)^2, \mathcal{O}_X)$ in this situation: $\mathcal{O}_X$ has a resolution of the form

$$0 \to \mathcal{O}_S\{x\}^{k-1} \xrightarrow{A} \mathcal{O}_S\{x\}^k \xrightarrow{\Delta} \mathcal{O}_X \to 0,$$

where Δ is given by the maximal minors of the matrix A, and a presentation of N_X is obtained from a $k \times (k^2-k)$-matrix C of $(k-2)$-minors of A by [Mar03, Proposition 8] (the argument is given there for space curves but directly generalizes to arbitrary Cohen-Macaulay singularities in codimension 2, noting that in the proof only the determinantal structure of the defining equations and the fact that X has codimension 2 are used). The same description holds for the corresponding objects A_0, C_0 at the special fibre, which in the end means that the columns of C resp. C_0 are generators for the syzygy modules of $\mathcal{O}_X^k/N_X$ resp. $\mathcal{O}_{X_0}^k/N_{X_0}$, so $\mathcal{O}_X^k/N_X$ is $\mathcal{O}_S$-flat and the condition of Corollary 2.5.4 is satisfied. Hence:

Corollary 2.5.5. *If $\xi : X \to S$ is a semi-universal deformation of an isolated singularity X_0 which is Cohen-Macaulay of codimension 2, then the modular stratum of X_0 equals the flattening stratum of $T^1(X/S)$.*

Analogous statement for deformations with section can be formulated as follows:

Proposition 2.5.6. *Let $\xi : X \to S$ be a semi-universal deformation of an isolated singularity X_0 with section $\sigma : S \to X$. Let X_0 be defined by $f = (f_1, \ldots, f_k) \in \mathbb{C}\{x\}^k$, and let X be given by $F = (F_1, \ldots, F_k) \in \mathcal{O}_S\{x\}^k$. Then a subgerm $M \subseteq S$ is modular if and only if $\tilde{T}^1(X/S, J_\sigma) \otimes_{\mathcal{O}_S} \mathcal{O}_M$ is a flat $\mathcal{O}_M$-module, where, as before, $J_\sigma := Ker(\sigma^*) \subseteq \mathcal{O}_X$.*

Proof. This follows easily from adapting the arguments in Proposition 2.5.2. ∎

Corollary 2.5.7. *If X_0 is an isolated complete intersection singularity, then the modular stratum (with section) of X_0 coincides with the flattening stratum of the relative Tjurina module $T^1(X/S, J_\sigma)$, where S is the base space of a semi-universal deformation $X \to S$ of X_0 and $J_\sigma = Ker(\sigma^*) \subseteq \mathcal{O}_X$ corresponds to the section $\sigma : S \to X$.*

Proof. X_0 being a complete intersection we have, for any subspace $M \subseteq S$, an exact sequence

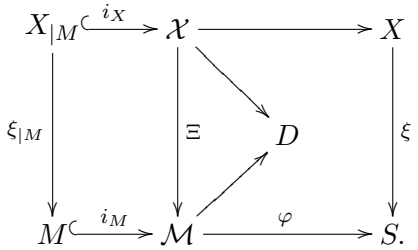
$$0 \to T^1(X/S, J_\sigma)_{|M} \longrightarrow \tilde{T}^1(X/S, J_\sigma)_{|M} \xrightarrow{r} \mathcal{O}_M^k \to 0$$

of $\mathcal{O}_M$-modules, where r is the residue map. Hence $T^1(X/S, J_\sigma)_{|M}$ is $\mathcal{O}_M$-flat if and only if $\tilde{T}^1(X/S, J_\sigma)_{|M}$ is. ∎

In fact, we can weaken the assumptions of the last corollaries to the more general condition $T^2(X_0) = 0$ (resp. $T^2(X_0, \mathfrak{m}_{X_0}) = 0$). We need the following lemma:

Lemma 2.5.8. *Let $\xi : X \to S$ be a semi-universal deformation (with section σ) of the singularity X_0 and $M \subseteq S$ a modular subspace. Then $T^1(\xi_{|M}) = 0$ (resp. $T^1(\xi_{|M}, (\mathcal{O}_M, J_{\sigma|M})) = 0$).*

Proof. We give a proof for the case of 'ordinary deformations'. For the functor $Def^s_{X_0}$ the statement follows by adding the given section at the necessary places. According to Theorem 1.2.7, elements of $T^1(\xi_{|M})$ correspond to deformations of the morphism $\xi_{|M}$ over the double point D. So let $\Xi : \mathcal{X} \to \mathcal{M}$ be a deformation of $\xi_{|M}$ over D. In particular, Ξ can be interpreted as a deformation of X_0 as well, hence by the versality of ξ there exists a morphism $\varphi : \mathcal{M} \to S$ inducing Ξ, i. e. we obtain the following diagram:



Since $M \subseteq S$ is modular, the composition $\varphi \circ i_M$ must be equal to the inclusion $M \hookrightarrow S$. Thus we can conclude that $\mathcal{M} \simeq M \times D$ and Ξ is isomorphic to the trivial deformation $\xi \times id_D : X \times D \to M \times D$, whence $T^1(\xi_{|M}) = 0$. ∎

Remark 2.5.9. This proof is a variant of the proof of [Pal90b, Proposition 2.5], where it has been shown that $T^1(\xi) = 0$ for a semi-universal deformation ξ of X_0 itself.

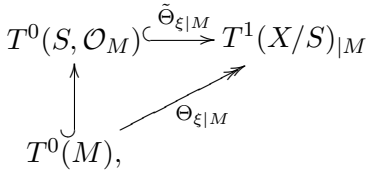
Using this fact we can now establish:

Proposition 2.5.10. *Let X_0 be an isolated singularity which is unobstructed, i. e. $T^2(X_0) = 0$. Let $\xi : X \to S$ be a semi-universal deformation of X_0, and $M \subseteq S$ a subgerm. Then M is modular if and only if $T^1(X/S)_{|M}$ is $\mathcal{O}_M$-flat.*

The analogous statement is true for deformations with section: If $\xi : X \to S$ is a semi-universal deformation with section σ of the unobstructed singularity X_0, then $M \subseteq S$ is modular if and only if $T^1(X/S, J_\sigma)_{|M}$ is $\mathcal{O}_M$-flat.

Proof. Since $T^1(X/S)_{|M}$ is finitely generated, its $\mathcal{O}_M$-flatness implies that it is a free $\mathcal{O}_M$-module of rank τ, the Tjurina number of X_0. Since $T^2(X_0) = 0$ we know that, in addition, $\mathcal{O}_S \simeq \mathbb{C}\{s_1, \ldots, s_\tau\}$ is smooth, whence $T^0(S, \mathcal{O}_M) \simeq \mathcal{O}_M^\tau$, and the Kodaira-Spencer map $\tilde{\Theta}_{\xi|M}$ sends free generators of $T^0(S, \mathcal{O}_M)$ to free generators of $T^1(X/S)_{|M}$. Thus $\tilde{\Theta}_{\xi|M}$ is an isomorphism, so in particular we can apply Criterion (2) in Theorem 2.2.4 and obtain that M is modular.

Conversely, assume $M \subseteq S$ is modular. We combine the relative Kodaira-Spencer maps of the deformations $\xi_{|M}$ and of the deformation ξ (now with coefficients in $(\mathcal{O}_M, \mathcal{O}_{X|M})$) into the single diagram (2.2.2):



where the injectivity of $\tilde{\Theta}_{\xi|M}$ follows from Theorem 2.2.4. Surjectivity of $\Theta_{\xi|M}$ is implied by the Kodaira-Spencer sequence

$$0 \longrightarrow T^0(X/S)_{|M} \longrightarrow T^0(\xi_{|M}) \longrightarrow T^0(M) \xrightarrow{\Theta_{\xi|M}} T^1(X/S)_{|M} \longrightarrow T^1(\xi_{|M}) \longrightarrow \cdots$$

of the deformation $\xi_{|M}$, combined with Lemma 2.5.8. It follows that $\mathcal{O}_M^\tau \simeq T^0(S, \mathcal{O}_M) \simeq T^1(X/S)_{|M}$ is a free and in particular a flat $\mathcal{O}_M$-module. ∎

Remark 2.5.11. The result above has been conjectured before, being known only for the cases of complete intersections and reduced space curves (with respect to Def_{X_0}). In particular it generalizes Corollaries 2.5.3 and 2.5.5 which form the main results of [Mar02a, section 6] and [Mar03].

Remark 2.5.12. Of course one may consider the flattening stratum of $T^1(X/S)$ for any deformation $\xi : X \to S$ of X_0. But if ξ is not a monodeformation, there is no connection to modular subspaces: As an example, take an arbitrary hypersurface singularity X_0 defined by $f \in \mathcal{O}_n$, and $g \in (f, J(f))$. Then $f + \varepsilon g$ gives a trivial deformation X_D of X_0 over the double point D, its Kodaira-Spencer map θ_ξ is the zero map, thus ξ is not a monodeformation, so D does not contain any modular space with respect to this deformation. On the other hand $T^1(X_D/D)$ is a flat $\mathcal{O}_D = \mathbb{C}[\varepsilon]$-module:

Indeed, let $rf + \sum_k r_k \frac{\partial f}{\partial x_k} = 0$ be a relation of $f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$. Then

$$r(f + \varepsilon g) + \sum_k r_k \frac{\partial}{\partial x_k}(f + \varepsilon g) = \varepsilon(rg + \sum_k r_k \frac{\partial g}{\partial x_k}),$$

and this again belongs to $\varepsilon(f, J(f))$. Namely, if $g = hf + \sum_i h_i \frac{\partial f}{\partial x_i}$ for suitable $h, h_1, \ldots, h_n \in \mathcal{O}_n$, we calculate:

$$\begin{aligned} rg + \sum_k r_k \frac{\partial g}{\partial x_k} &= \sum_k r_k \sum_i h_i \frac{\partial^2 f}{\partial x_i \partial x_k} && \mod(f, J(f)) \\ &= \sum_i h_i \frac{\partial}{\partial x_i}\left(\sum_k r_k \frac{\partial f}{\partial x_k}\right) - \sum_i h_i \sum_k \frac{\partial r_k}{\partial x_i}\frac{\partial f}{\partial x_k} \\ &= -\sum_i h_i \frac{\partial rf}{\partial x_i} = 0 && \mod(f, J(f). \end{aligned}$$

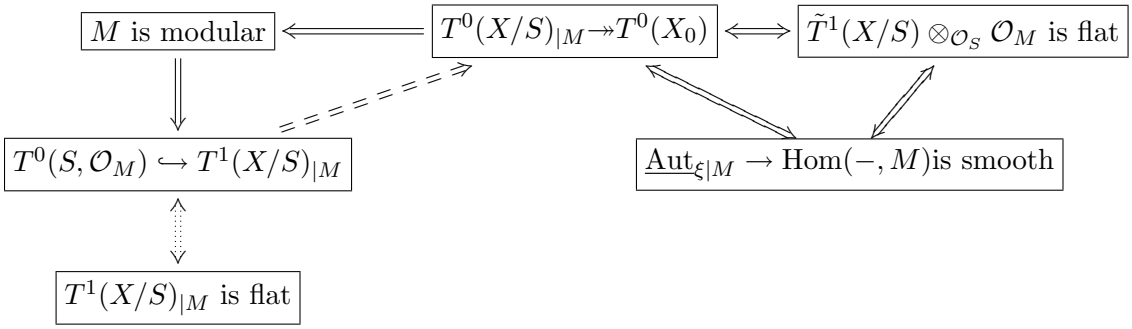
Thus $rg + \sum_k r_k \frac{\partial g}{\partial x_k}$ is equal to $pf + \sum_k p_k \frac{\partial f}{\partial x_k}$ for some $p, p_1, \ldots, p_n \in \mathcal{O}_n$, so

$$(r - \varepsilon p, r_1 - \varepsilon p_1, \ldots, r_n - \varepsilon p_n)$$

lifts the given relation to $\mathcal{O}_n[\varepsilon]$. This shows that for any trivial first-order infinitesimal deformation of a hypersurface, $T^1(X_D/D)$ is $\mathcal{O}_D$-flat.

As a concrete example take, for instance, $f := x^4 + y^7 \in \mathbb{C}\{x, y\}$ and $g := x^3$. The only non-trivial relation that we have to consider is the Euler relation $1 \cdot f - \frac{1}{4}x\frac{\partial f}{\partial x} - \frac{1}{7}y\frac{\partial f}{\partial y} = 0$, which yields $g - \frac{3}{4}g = 0 \bmod (f, J(f))$ when putting g in.

Let us summarize the various criteria for a germ $M \subseteq S$ to be modular, $\xi : X \to S$ a monodeformation of the singularity X_0. The assertions of Proposition 2.1.9, Theorem 2.2.4, Proposition 2.5.2 and Proposition 2.5.10 can be combined into the following diagram:



The dashed arrow only holds in case ξ is in fact semi-universal, so in this case all assertions are equivalent, otherwise the statements on the right hand side only give *sufficient* criteria for M to be modular. Furthermore, the dotted one holds in case $T^2(X_0) = 0$ by Proposition 2.5.10. Similarly, we have derived the same relationship between the corresponding statements with regards to the functor of deformations with section.

These characterizations of modularity of $M \subseteq S$ as flatness of a suitable $\mathcal{O}_M$-module make it possible to compute the modular stratum of X_0, at least up to a given order. It will be the purpose of the subsequent chapter three to demonstrate how these theoretical criteria can be turned into an algorithm, and to describe the resulting algorithm and its implementation in SINGULAR, making it possible to compute non-trivial examples of modular strata of singularities.

2.5.2 Comparing flatness conditions

We close this section by some remarks on the interplay of the modular strata with respect to both deformation functors under consideration. As already noted after the proof of Theorem 2.2.5, the modular strata themselves cannot be compared directly, because they are subgerms of the different semi-universal deformations. However, we can at least discuss some connection between modular subspaces for both functors. First of all note the following straightforward fact:

Lemma 2.5.13. *Let $\xi : X \to S$ be a deformation of X_0 with section $\sigma : S \to X$. If $M \subseteq S$ is modular with respect to the functor Def_{X_0} (i. e. considering ξ as an 'ordinary' deformation), then it is modular with respect to $Def^s_{X_0}$, too.*

Proof. Suppose M is Def_{X_0}-modular. Let $\varphi : T \to M$ and $\psi : T \to S$ be morphisms such that $\varphi^*(\xi_{|M})$ and $\psi^*(\xi)$ are isomorphic deformations with section of X_0. Then they also give the same element of $Def_{X_0}(S)$, hence $\varphi = \psi$ by assumption. Thus M is $Def^s_{X_0}$-modular, too. ∎

We now specialize down to hypersurfaces and restrict ourselves to deformations with singular section, as introduced in section 1.3. We can interpret such deformations as deformations leaving the embedding dimension constant, the terms of its defining equation then belong $\mathfrak{m}^2$ – but this is only true for hypersurfaces, so we have to restrict ourselves in the next statement to singularities of codimension one:

Proposition 2.5.14. *Let $\xi : X \to S$ be a deformation with singular section of a hypersurface X_0, and let $M \subseteq S$. Assume additionally that S is a smooth germ. Then $T^1(X/S)_{|M}$ is a flat $\mathcal{O}_M$-module if and only if so is $T^1(X/S, J_\sigma)_{|M}$.*

Proof. The short exact sequence of $\mathcal{O}_{X_{|M}}$-modules

$$0 \to J_{\sigma|M} \to \mathcal{O}_{X_{|M}} \to \mathcal{O}_{X_{|M}}/J_{\sigma|M} \simeq \mathcal{O}_M \to 0$$

induces the long exact cohomology sequence

$$0 \longrightarrow T^0(X/S, J_\sigma)_{|M} \xrightarrow{\iota_0} T^0(X/S)_{|M} \xrightarrow{\pi_0} T^0(X/S, \mathcal{O}_S)_{|M} \xrightarrow{\delta_0}$$
$$\xrightarrow{\delta_0} T^1(X/S, J_\sigma)_{|M} \xrightarrow{\iota_1} T^1(X/S)_{|M} \xrightarrow{\pi_1} T^1(X/S, \mathcal{O}_S)_{|M} \longrightarrow 0. \quad (2.5.1)$$

Let $f \in \mathcal{O}_n$ define X_0. Since $X_{|M} \to M$ is a deformation with singular section of X_0, we may assume that $X_{|M}$ is defined by $F \in (x)^2\mathcal{O}_M\{x\}$. In this situation we have $T^1(X/S, \mathcal{O}_S)_{|M} \simeq \mathcal{O}_M$ since it is the cokernel of

$$\mathrm{Der}_{\mathcal{O}_M}(\mathcal{O}_M\{x\}, \mathcal{O}_M) \to \mathrm{Hom}_{\mathcal{O}_{X|M}}((F)/(F^2), \mathcal{O}_M) \simeq \mathcal{O}_M,$$
$$\delta \mapsto \delta(F)$$

which is the zero map as $F \in (x)^2\mathcal{O}_M\{x\}$. In addition, $Ker(\iota_1)$ is the $\mathcal{O}_M$-module generated by $\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}$ modulo $(F) + (x_j \frac{\partial F}{\partial x_i})_{i,j}$. But this is a free $\mathcal{O}_M$-module: namely, suppose $\sum_i G_i(s)\frac{\partial F}{\partial x_i} = H \cdot F + \sum_{i,j} H_{ij}x_j\frac{\partial F}{\partial x_i}$. Since, by the smoothness assumption on S, $(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$ form a regular sequence, this implies that, for all i, $G_i(s) - \sum_j H_{ij}x_j \in (F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$, i. e.

$$G_i(s) = \sum\nolimits_j H_{ij}x_j + A \cdot F + \sum\nolimits_k B_k \frac{\partial F}{\partial x_k}$$

for some $A, B_k \in \mathcal{O}_M\{x\}$. But this is only possible if $G_i = 0$ since $F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \in (x)\mathcal{O}_M\{x\}$.

Thus the long exact sequence (2.5.1) can be split into two short exact sequences

$$0 \to Ker(\iota_1) \simeq \mathcal{O}_M^n \to T^1(X/S, J_\sigma)_{|M} \to Im(\iota_1) \to 0 \quad (*)$$

and

$$0 \to Ker(\pi_1) \to T^1(X/S)_{|M} \to \mathcal{O}_M \to 0,$$

from which we derive that if $T^1(X/S)_{|M}$ is $\mathcal{O}_M$-flat, then so is $T^1(X/S, J_\sigma)_{|M}$.

On the other hand, suppose $T^1(X/S, J_\sigma)_{|M}$ is a flat $\mathcal{O}_M$-module. Hence it is free since it is finitely generated over $\mathcal{O}_M$. But, by what was said before, this module contains the free submodule generated by $\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}$. Extend these elements to a system of generators of $T^1(X/S, J_\sigma)_{|M}$. Modulo $\mathfrak{m}_M$, they generate the $\mathbb{C}$-vector space $T^1(X_0, \mathfrak{m}_{X_0})$, and since $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ are $\mathbb{C}$-linearly independent modulo $(x)J(f)$ by Corollary 1.3.5, we may take a basis containing all $\frac{\partial f}{\partial x_i}$ which means that we find a free $\mathcal{O}_M$-basis of $T^1(X/S, J_\sigma)_{|M}$ containing all $\frac{\partial F}{\partial x_i}$. But this implies that the sequence $(*)$ is split exact, hence $Im(\iota_1)$ is also flat which in the end establishes the flatness of $T^1(X/S)_{|M}$ using the lower exact sequence from above. ∎

Nevertheless, even when we restrict ourselves to a deformation $\xi : X \to S$ with singular section σ of the hypersurface X_0 which is a monodeformation for Def_{X_0} (and hence for $Def^s_{X_0}$, too), we cannot conclude from this result that the maximal modular subspaces (with respect to both functors) inside this deformation coincide, since we have only proved the following implications for a subgerm $M \subseteq S$ and only for the case that S is smooth:

$$\begin{array}{ccc} M \text{ is } Def_{X_0}\text{-modular} & \Longrightarrow & M \text{ is } Def^s_{X_0}\text{-modular} \\ \Uparrow & & \Uparrow \\ T^1(X/S)_{|M} \text{ is } \mathcal{O}_M\text{-flat} & \Longleftrightarrow & T^1(X/S, J_\sigma)_{|M} \text{ is } \mathcal{O}_M\text{-flat.} \end{array}$$

Nevertheless, computations make it plausible that in fact there should hold equivalence in the first row, and we finish this chapter by the following

Conjecture 2.5.15. Let X_0 be an isolated singularity and $\xi : X \to S$ a deformation of X_0 with section $\sigma : S \to X$ such that ξ is a monodeformation with respect to both Def_{X_0} and to $Def^s_{X_0}$. Then $M \subseteq S$ is Def_{X_0}-modular if and only if it is $Def^s_{X_0}$-modular.

Chapter 3

Computing modular strata: algorithm and implementation

Although the notion of modular deformations appeared in the literature already about twenty-five years ago, there is only little knowledge regarding explicit examples of modular strata for singularities: The modular stratum of quasihomogeneous complete intersections in positive dimension was determined by Alexandrov in [Ale85], the result has been stated in Proposition 2.4.2 and extended to the functor of deformations with section. Besides, it is only known that in many cases the modular stratum of a singularity X_0 is an Artinian germ. Some examples are given by Palamodov in [Pal90a], [Pal94] – others are obtained from the $T_{p,q,r}$-series singularities defined by

$$f = x^p + y^q + z^r + \lambda \cdot xyz \in \mathbb{C}\{x, y, z\}, \qquad \tfrac{1}{p} + \tfrac{1}{q} + \tfrac{1}{r} \leq 1,$$

which have an Artinian modular stratum for all but finitely many (p, q, r) (we will make this precise later in section 4.2).

The problem in deriving general results about the modular stratum of a singularity is that both criteria that one has at hand – injectivity of the relative Kodaira-Spencer map $\tilde{\Theta}_{\xi|M}$ and surjectivity of $T^0(X_{|M}/M) \to T^0(X_0)$ – require to determine objects which in general are quite hard to describe. In particular, there are not many general results concerning the structure of the module of derivations $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X_0})$ for non-quasihomogeneous singularities. Our goal is therefore to use the theoretical background developed in the previous chapter and to turn it into an algorithm that makes it possible to calculate (at least jets of) the modular stratum of an isolated singularity by means of a computer, and we also describe in some detail how it has been implemented in the computer algebra system Singular [GPS02] as library `modular.lib` [HM04]. This work is built upon the algorithm by B. Martin that computes local flattenings of modules over local rings [Mar02a] and on a first implementation of this algorithm in Singular.

Thus this chapter is devoted to explaining how this algorithm works – in particular we show that it gives the correct result for any isolated singularity (although, in practice, Singular will not be able to compute higher jets of the modular stratum of more complicated singularities due to the complexity of the computations involved). For reasons of simplicity and clarity, we restrict our presentation of the algorithm to the functor Def_{X_0}. Obvious changes apply to deformations with section, just switch to the modules $T^i(X_0, \mathfrak{m}_{X_0})$ instead of $T^i(X_0)$ etc. Using this algorithm and its implementation, we will calculate in the last chapter

four a series of non-trivial examples of modular strata of singularities, exhibiting phenomena most of which have not been presented before.

We do not try to conceal that, in fact, it will not be possible to compute higher order jets of the modular stratum when the defining equations of the singularity become more complicated – the data and computations involved are just to demanding in terms of memory usage and computing time. On the other hand, despite this drawback we are able to describe a relatively efficient way of computing the tangent space $T(M_{X_0})$ to the modular stratum, and thus we can at least obtain equations for the modular family in many interesting cases.

3.1 An algorithm to compute the modular stratum

To start with, let be given an isolated singularity X_0, defined by $f_1, \ldots, f_k \in \mathbb{C}\{x_1, \ldots, x_n\}$, and $\xi : X \to S$ a semi-universal deformation of X_0, given by $F_1, \ldots, F_k \in \mathcal{O}_S\{x\}$ such that $F_i = f_i \bmod \mathfrak{m}_S\mathcal{O}_S\{x\}$ for all i and $\mathcal{O}_S = \mathbb{C}\{s_1, \ldots, s_\tau\}/I$, $\tau = \tau(X_0)$ the Tjurina number of X_0.

Recall the following constructions and results of chapter two:

- Proposition 2.2.10: The tangent space to the modular stratum $M_{X_0} \subseteq S$ is given as

$$T(M_{X_0}) = \left\{t \in T(S) : [\delta, t] = 0 \text{ for all } \delta \in T^0(X_0)\right\},$$

  where we identify $T(S)$ and $T^1(X_0)$ by means of the Kodaira-Spencer map θ_ξ, and $[-,-] : T^0(X_0) \times T^1(X_0) \to T^1(X_0)$ is the Lie bracket in cotangent cohomology.

- Definition 2.2.8 and Proposition 2.2.9: If $A \subseteq S$ is an Artinian modular subgerm and $B \subseteq S$ is obtained from A by a small extension $\eta : 0 \to J \to \mathcal{O}_B \to \mathcal{O}_A \to 0$, then B is modular if and only if $ob_{\xi,\eta} \in \mathrm{Hom}(T^0(X_0), T^1(X_0)) \otimes J$ vanishes.

- $M \subseteq S$ is modular if and only if $\tilde{T}^1(X/S)_{|M} = \mathcal{O}_M\{x\}^k/P_{F|M}$ is a flat $\mathcal{O}_M$-module, and this module is the quotient of $\mathcal{O}_M\{x\}^k$ by the matrix $P_{F|M}$ introduced in section 2.5.

The idea is now the following: First compute the tangent space to the modular stratum. That is, we determine the minimal ideal $J_{(1)} \subseteq \mathfrak{m}_S$ such that the restriction of ξ to the subspace of S defined by $J_{(1)} + \mathfrak{m}_S^2$ is modular. Afterwards we obtain the approximation in order k of the ideal of the modular stratum order by order, by killing obstructions – or, from a different point of view, preserving flatness of the module $\tilde{T}^1$.

3.1.1 Computing the tangent space

In order to determine $T(M_{X_0})$ we need an explicit description of $T^0(X_0)$, $T^1(X_0)$ and the Lie bracket $T^0 \times T^1 \to T^1$. We obtain the data of these two modules and this map as follows:

- Consider the matrix

$$P_f := \begin{pmatrix} f_1 & & f_2 & & \cdots & f_k & & \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \ddots & & \ddots & & & \ddots & \vdots & & \vdots \\ & & f_1 & & f_2 & \cdots & & f_k & \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix} \tag{3.1.1}$$

already used in order to provide the flatness criteria for modularity in section 2.5. If $(h_{11},\ldots,h_{k1},h_{12},\ldots,h_{kk},-g_1,\ldots,-g_n)$ is a syzygy of its columns, then we can define $\delta := \sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j} \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_n)$ which represents a derivation in $T^0(X_0)$. Hence computing the first syzygy module of the columns of P_f yields generators $r_1,\ldots,r_m$ of $syz(P_f)$ inducing representatives $\delta_1,\ldots,\delta_m$ of generators of $T^0(X_0)$, together with $k\times k$-matrices $h_1,\ldots h_m$, such that

$$\begin{pmatrix} \delta_i(f_1) \\ \vdots \\ \delta_i(f_k) \end{pmatrix} = h_i \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix}$$

for all $i = 1,\ldots,m$.

- Let us give a short explanation of how one obtains a $\mathbb{C}$-vector space basis of $T^1(X_0)$ (for details, see [Ste03] or [Mar99], where an explicit description of the necessary steps in SINGULAR is included): $T^1(X_0)$ is the quotient of the normal module $N_{X_0} = \mathrm{Hom}_{\mathcal{O}_{X_0}}((f)/(f)^2,\mathcal{O}_{X_0}) = \mathrm{Hom}_{\mathcal{O}_n}((f),\mathcal{O}_{X_0})$ by the submodule generated by (classes of) the columns of the Jacobian $J(f)$ of f. Let S be a $k\times l$-matrix of generators of the first syzygy module of $f_1,\ldots,f_k$, cf. Remark 1.1.15. Then N_{X_0} can be identified with the first syzygy module of $\bar{S}^t$, where the bar indicates classes in $\mathcal{O}_{X_0}$. Consider a free resolution of $Coker(\bar{S}^t)$:

$$0 \longleftarrow Coker(\bar{S}^t) \longleftarrow \mathcal{O}_{X_0}^{l} \xleftarrow{\bar{S}^t} \mathcal{O}_{X_0}^{k} \xleftarrow{N_1} \mathcal{O}_{X_0}^{k_2} \xleftarrow{N_2} \mathcal{O}_{X_0}^{k_3}$$
$$\uparrow{\scriptstyle \overline{J(f)}} \quad \nearrow{\scriptstyle J'}$$
$$\mathcal{O}_{X_0}^{n}$$

  Then $N_{X_0} = Ker(\bar{S}^t) = Im(N_1)$, hence $T^1(X_0) = Im(N_1)/Im(\overline{J(f)}) = Coker(N_2 \oplus J')$. Thus we have to find a lift of $\overline{J(f)}$ to $\mathcal{O}_{X_0}^{k_2}$ by means of N_1, and then a $\mathbb{C}$-basis of $T^1(X_0)$ is given by multiplying a basis of the quotient $\mathcal{O}_{X_0}^{k_2}/(N_2 \oplus J')$ by N_1.

- So let us assume we have determined a $\mathbb{C}$-basis $b_1,\ldots,b_\tau$ of $T^1(X_0)$ (strictly speaking we refer to the b_i as elements of $\mathbb{C}\{x\}^k$ whose classes induce a basis of $T^1(X_0)$), in particular we assume that the chosen semi-universal deformation has the property that $\theta_\xi(s_i) = b_i$, where $s_1,\ldots,s_\tau$ generate the maximal ideal of $\mathcal{O}_S$.

- For each $i = 1,\ldots,\tau$ and $j = 1,\ldots,m$ we now determine the Lie bracket of δ_j and b_i by computing $l_{ijk} \in \mathbb{C}$ such that

$$[\delta_j, b_i] = \overline{\delta_j(b_i) - h_j \cdot b_i} = \sum_{k=1}^{\tau} l_{ijk} b_k,$$

  where the bar means taking the class in $T^1(X_0)$; and we write shortly $\delta_j(b_i)$ for the component-wise application of δ_j to the entries of b_i. Arranging the l_{ijk} in the following manner we end up with a total description of the bracket $T^0 \times T^1 \to T^1$ contained in the $\tau \times \tau m$-matrix (whose entries are complex numbers)

$$B := \begin{pmatrix} l_{111} & \cdots & l_{11\tau} & l_{121} & \cdots & l_{12\tau} & \cdots & l_{1m1} & \cdots & l_{1m\tau} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ l_{\tau 11} & \cdots & l_{\tau 1\tau} & l_{\tau 21} & \cdots & l_{\tau 2\tau} & \cdots & l_{\tau m1} & \cdots & l_{\tau m\tau} \end{pmatrix} \in Mat(\tau, \tau m; \mathbb{C}).$$

Knowing this matrix B it is now easy to determine $T(M_{X_0})$ by simple linear algebra: Equations for $T(M_{X_0})$ are given by the conditions

$$0 = [\delta_j, \sum_i c_i \theta_\xi(s_i)] = \sum_i c_i [\delta_j, b_i] = \sum_i c_i \sum_k l_{ijk} b_k = \sum_k \left(\sum_i l_{ijk} c_i \right) b_k$$

for $j = 1, \ldots, m$, i. e. the tangent space is given by all those $\sum_i c_i s_i$ such that $(c_1, \ldots, c_\tau) \cdot B = 0$. Let C be an $e \times \tau$-matrix $(c_{ij}) \in Mat(e, \tau, \mathbb{C})$, whose rows form a $\mathbb{C}$-vector space basis of the annihilator of the column space of B. We now restrict the base $(S, 0) \subseteq (\mathbb{C}^\tau, 0)$ to this linear subspace, which is achieved by the algebra map

$$\begin{aligned} \varphi : \mathbb{C}\{s_1, \ldots, s_\tau\}/I \twoheadrightarrow & \mathbb{C}\{t_1, \ldots, t_e\}/(t)^2 \qquad (*) \\ s_i \mapsto & \sum_{j=1}^{e} c_{ji} t_j, \end{aligned}$$

inducing a modular subdeformation $\varphi^*(\xi) : X_T \to T$ of ξ.

We have determined the tangent space directly in such a manner that we can reduce as many variables as possible, which is crucial in terms of computational complexity: Of course, we could have computed a minimal ideal $J_{(1)} \subseteq \mathfrak{m}_S$ of linear forms in s defining the tangent space, but the advantage is that explicitly computing the tangent space of M_{X_0} by means of the map in $(*)$ may push down the embedding dimension of the basis considerably. In particular, getting φ for granted out of this calculation, we can easily compute the equations $F(x, \varphi(s)) \in \mathcal{O}_T\{x\}^k$ defining $\varphi^*(\xi)$ out of the equations F for ξ. Since this map results quite naturally from the above computation, it also seems more efficiently to do this directly than to compute equations of $J_{(1)}$ and then perform the appropriate coordinate change resp. projection in a second step, as had been done in first implementations.*

As an example, consider the plane curve singularity given by $f := x^{10} + x^8 y^3 + x^5 y^7 + y^{12} \in \mathbb{C}\{x, y\}$. Its Tjurina number is 83 and the embedding dimension of its modular stratum is 30. This number has been computed by SINGULAR in approx. 13 seconds† using our implementation but does not finish when computing using the other method indicated in the previous paragraph. In fact, most of the time is used for finding lifts of the δ_i – the computation of the embedding dimension itself only takes approx. two seconds.

In particular, the above construction gives an explicit formula for the embedding dimension of M_{X_0} in terms of the cotangent cohomology of X_0:

Proposition 3.1.1. *Let X_0 be an isolated singularity, and let $\delta_1, \ldots, \delta_m$ be generators of $T^0(X_0)$ as an $\mathcal{O}_{X_0}$-module, and $b_1, \ldots, b_\tau$ be a $\mathbb{C}$-basis of $T^1(X_0)$. If the matrix $B = (l_{ijk}) \in Mat(\tau, m\tau, \mathbb{C})$ from above represents the Lie bracket $[-,-] : T^0(X_0) \times T^1(X_0) \to T^1(X_0)$, then*

$$\text{e.dim}(M_{X_0}) = \tau - \text{rank}(B).$$

*In fact, our approach means geometrically that we are restricting the computation of the modular stratum M_{X_0} to the tangent subspace of $M_{X_0} \subseteq (\mathbb{C}^\tau, 0)$. For many examples this is feasible but actually this needs not give the full modular stratum in general. Therefore, when one is interested in a complete answer regarding the dimension or the equations of the modular stratum, one has to compute with respect to the given embedding in $(\mathbb{C}^\tau, 0)$, i. e. not reducing the number of variables in the way described above.

†All computations and timings mentioned were performed on an Intel Pentium III, 933 MHz machine with 256 MBytes of memory, running Red Hat Linux 9 and SINGULAR revision 2-0-4.

Remark 3.1.2. Alternatively, one could use the finite-dimensional $\mathbb{C}$-vector space $T^0_\bullet(X_0)$ instead of $T^0(X_0)$ (which in fact has dimension τ for complete intersections by Proposition 2.3.2). Our implementation uses $T^0(X_0)$ for some reasons originating in the way the computations have to be done in SINGULAR, as the second of the following examples shall demonstrate.

Example 3.1.3.

(1) As a first example, we consider a quasihomogeneous singularity, where we already know the result from Proposition 2.4.2. Take, for instance, $f := x^4 + y^8 \in \mathbb{C}\{x, y\}$. Its Tjurina number τ is 21, a $\mathbb{C}$-basis of $T^1(X_0)$ is induced by $\{1, x, x^2, y, xy, x^2y, \ldots, x^2y^6\}$. Generators of $T^0(X_0)$ are given by the Euler derivation $\delta_E := 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ (we take a multiple such that we get $\delta_E(f) = 8f$ for the sake of simplicity) and the trivial Koszul relation $\delta_K = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}$, which we do not need consider. Now $\delta_E(x^py^q) - 8x^py^q = (2p + q - 8)x^py^q$, so for this example, B is a $\tau \times 2\tau$-block matrix:

- Its first $\tau \times \tau$-block is a diagonal matrix with diagonal entry $2p + q - 8$ at the position corresponding to the basis monomial x^py^q, and this gives zero precisely for $(p, q) = (1, 6)$ and $(2, 4)$.
- Besides, for δ_K we have an additional $\tau \times \tau$-block of zeros.

Thus the embedding dimension of the modular stratum is two, and the corresponding family is given as

$$f + t_1xy^6 + t_2x^2y^4$$

over $\mathbb{C}\{t_1, t_2\}/(t_1, t_2)^2$, agreeing with Proposition 2.4.2, which also tells us that this already yields the maximal modular family over the smooth base with local ring $\mathbb{C}\{t_1, t_2\}$.

(2) As an example of a non-quasihomogeneous singularity, take $f := \frac{1}{4}x_1^4 + \frac{1}{3}x_2^3 + \frac{1}{3}x_3^3 + x_1x_2x_3 \in \mathbb{C}\{x_1, x_2, x_3\}$, which defines a $T_{4,3,3}$-singularity. In this situation, a SINGULAR-computation yields six generators of $syz(f, J(f))$. Alternatively, note that in this case $(J(f) : f) = \mathfrak{m}$, the maximal ideal in $\mathbb{C}\{x_1, x_2, x_3\}$, as $\mu(f) - \tau(f) = 1$ is the dimension of $T^0_\bullet$ which is isomorphic to the kernel of the multiplication map $Q(f) \xrightarrow{\cdot f} Q(f)$ in the local algebra of f by Proposition 2.3.2. So it suffices to determine three derivations δ_i such that $\delta_i(f) = x_if$ for $i = 1, 2, 3$, and to continue as above, with only three derivations to be taken into account instead of six ones, i. e. cutting down the number of partial computations by one half. Computing them in SINGULAR yields:

$$\begin{aligned}\left(\tfrac{1}{12} + \tfrac{1}{12}x_1\right)x_1f = &\left(\tfrac{1}{48}x_1^2 + \tfrac{1}{144}x_2x_3 + \tfrac{1}{48}x_1x_2x_3 + \tfrac{1}{144}x_3^3\right)\tfrac{\partial f}{\partial x_1}\\ &+ \left(-\tfrac{5}{144}x_3^2 - \tfrac{1}{36}x_1x_3^2\right)\tfrac{\partial f}{\partial x_2}\\ &+ \left(\tfrac{1}{36}x_2^2 + \tfrac{1}{16}x_1x_3 + \tfrac{1}{36}x_1x_2^2 + \tfrac{1}{18}x_1^2x_3 - \tfrac{1}{144}x_2x_3^2 - \tfrac{1}{144}x_1^3x_3\right)\tfrac{\partial f}{\partial x_3}\end{aligned}$$

for δ_1 and similarly for δ_2, δ_3. In any case, we can only compute $\delta_i(f) = u_ix_if$, where $u_i \in \mathbb{C}\{x_1, x_2, x_3\}$ is a unit, which makes further steps much more complicated when trying to implement them in SINGULAR. The reason why these units necessarily pop up comes from the fact that, over polynomial rings with local orderings (what is the framework within which all these computations have to be performed in SINGULAR), there exists only a *weak normal form*, i. e. if $g \in R^k$ is an element of the submodule generated

by $m_1, \ldots, m_s \in R^k$, then we can only compute a representation $ug = \sum_j c_j m_j$ with $c_j \in R$ and $u \in R^*$ – for a detailed discussion, see [GP02, chapters 1–2], for example.

Let us come back to our example. The result for the tangent space to the modular stratum is the following: Take $\{1, x_1, x_1^2, x_1^3, x_2, x_2^2, x_3, x_3^2\}$ as $\mathbb{C}$-basis of $T^1(X_0)$, then $[\delta_1, x_i^k] = \overline{\delta_1(x_i^k) - u_1 x_1 \cdot x_i^k} = 0$ for $i = 2, 3$, but $[\delta_1, x_1^k] = \left(\frac{k}{48} - \frac{1}{12}\right) x_1^{k+1}$ modulo $(f, J(f))$, so this only gives zero in $T^1(X_0)$ for $k = 3$. For δ_2, δ_3 we obtain similar results, and we end up with the modular family given by

$$f + t_1 x_1^3 + t_2 x_2^2 + t_3 x_3^2$$

over $\mathbb{C}\{t_1, t_2, t_3\}/(t_1, t_2, t_3)^2$ as first-order approximation of the modular stratum. In particular its embedding dimension equals three – we will see how this can be generalized to arbitrary singularities of the T-series in section 4.2.

3.1.2 Lifting obstructions to higher order

Having constructed $T(M_{X_0})$, we use the obstruction calculus developed in section 2.2 in order to find higher order jets of the modular stratum, more precisely we use its interpretation as the condition of preserving flatness of the module $\tilde{T}^1(X/S)$: Given a modular subgerm $A \subseteq S$ and a small extension $A \subset B$ we want to find the maximal subgerm between A and B preserving modularity. We know by the modularity of A that we can lift all vector fields in $T^0(X_0)$ to $T^0(X/S)_{|A}$ and we want do determine the maximal extension $B' \subseteq B$ such that they can actually be lifted to $T^0(X/S)_{|B'}$. In terms of Proposition 2.5.2, this means that we want to find the maximal extension B' such that $\tilde{T}^1(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{B'}$ is still $\mathcal{O}_{B'}$-flat.

We now have to introduce some more notations in order to describe these conditions in terms of the equations of the given singularity. To this end, let, as before

$$P_F := \begin{pmatrix} F_1 & & & F_2 & & & \cdots & F_k & & & \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ & \ddots & & & \ddots & & \cdots & & \ddots & & \vdots & & \vdots \\ & & F_1 & & & F_2 & \cdots & & & F_k & \frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_n} \end{pmatrix},$$

and let $P_{F,e}$ be its e-th graded component in (s), and $P_{F,(e)} := \sum_{i=0}^{e} P_{F,i}$. Suppose we have already computed $J_{(e)} \subseteq \mathfrak{m}_S^e$ such that the subgerm defined by $J_{(e)} + \mathfrak{m}_S^{e+1}$ is modular, and this is the minimal ideal in $\mathfrak{m}_S^{e+1}$ with this property. Let $\delta_1, \ldots, \delta_m$ be elements of $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_n)$ inducing generators of $T^0(X_0)$ (i. e. $\delta_i(f) = h_i \cdot f$ for suitable $h_i \in Mat(k, k; \mathcal{O}_n)$, and their coefficients correspond to syzygies $r_1, \ldots, r_m$ of P_f). We can then find lifts $\delta_1^{(e)}, \ldots, \delta_m^{(e)}$ and $H_1^{(e)}, \ldots, H_m^{(e)}$ which correspond to lifts $R_1^{(e)}, \ldots, R_m^{(e)}$ of the syzygies r_i of P_f to syzygies of $P_{F,(e)}$, and again we denote by R_j^i the i-th graded component in (s).

This means that, writing $R^{(e)}$ for the matrix with columns the $R_j^{(e)}$, we have $P_{F,(e)} \cdot R^{(e)} = 0$ modulo $J_{(e)} + \mathfrak{m}_S^{e+1}$. Now we compute the obstructions with respect to the small extension

$$0 \to (J_{(e)} + \mathfrak{m}_S^{e+1})/\mathfrak{m}_S(J_{(e)} + \mathfrak{m}_S^{e+1}) \to \mathcal{O}_S/\mathfrak{m}_S(J_{(e)} + \mathfrak{m}_S^{e+1}) \to \mathcal{O}_S/(J_{(e)} + \mathfrak{m}_S^{e+1}) \to 0.$$

To this end, the obstruction for lifting δ_j is given by the class of

$$P_{F,(e+1)} R_j^{(e)} = P_{F,1} R_j^e + \ldots + P_{F,e+1} R_j^0 \bmod \mathfrak{m}_S(J_{(e)} + \mathfrak{m}_S^{e+1})$$

in $T^1(X_0) \otimes (J_{(e)} + \mathfrak{m}_S^{e+1})/\mathfrak{m}_S(J_{(e)} + \mathfrak{m}_S^{e+1})$, i. e. reducing modulo $P_f = P_{F,0}$ we find

$$P_{F,(e+1)}R_j^{(e)} + P_{F,0}R_j^{e+1} = \sum_{i=1}^{\tau} w_{ij}b_i$$

for (classes of) some elements $w_{ij} \in \mathcal{O}_S$, homogeneous of degree $e+1$. Doing this for all $\delta_1, \ldots, \delta_m$, we obtain elements w_{ij} forming the degree-$(e+1)$-part of $J_{(e+1)}$, the maximal modular extension to order $e+1$.

Example 3.1.4. Let us calculate explicitly what this means for the $T_{4,3,3}$-singularity whose tangent space had been computed in Example 3.1.3 (2): It is defined by $f = \frac{1}{4}x_1^4 + \frac{1}{3}x_2^3 + \frac{1}{3}x_3^3 + x_1x_2x_3 \in \mathbb{C}\{x_1, x_2, x_3\}$, and the first-order approximation of the modular stratum had been calculated as $(t_1, t_2, t_3)^2 \subseteq \mathbb{C}\{t_1, t_2, t_3\}$, with corresponding modular family $f + t_1x_1^3 + t_2x_2^2 + t_3x_3^2$. In this case, P_f equals

$$\left(\tfrac{1}{4}x_1^4 + \tfrac{1}{3}x_2^3 + \tfrac{1}{3}x_3^3 + x_1x_2x_3 \quad x_1^3 + x_2x_3 \quad x_2^2 + x_1x_3 \quad x_3^2 + x_1x_2\right),$$

and a matrix $R^{(0)}$ of generators of $syz(P_f)$ is given by

$$\begin{pmatrix}
12x_1{+}12x_2x_3 & 12x_2{-}12x_3^2 & 12x_3{+}12x_1x_3 & 0 & 0 & 0 \\
-3x_1^2{-}x_2x_3{-}4x_1x_2x_3{-}x_3^3 & -4x_1x_2{+}3x_1x_3^2 & -4x_1x_3{-}3x_1^2x_3 & 0 & x_1x_2{+}x_3^2 & x_2^2{+}x_1x_3 \\
-4x_1x_2{+}x_3^2{-}4x_2^2x_3{-}x_1x_3^2 & -4x_2^2{-}x_1^2x_3{+}4x_3^2 & -4x_2x_3{+}x_1^3{-}4x_1x_2x_3 & x_1x_2{+}x_3^2 & 0 & -x_2x_3{-}x_1^3 \\
-5x_1x_3{+}x_1^2x_3{-}3x_2x_3^2{+}x_1^3x_3 & -4x_2x_3{+}x_1^3{+}x_1x_2x_3{+}4x_3^3 & -4x_3^2{-}x_1^2x_2{-}4x_1x_3^2 & -x_2^2{-}x_1x_3 & -x_2x_3{-}x_1^3 & 0
\end{pmatrix},$$

illustrating how complicated already the first step of computing these syzygies becomes even for such a relatively simple example. In fact, computing `syz(`P_f`)` in SINGULAR yields one more column, the next to last one is doubled. Since the last columns (with entry 0 in the first row) correspond to trivial derivations, we could omit them as well. We keep them here in order to demonstrate the steps and intermediate results produced by SINGULAR.

Nevertheless, we can compute a lift $R^{(1)}$ such that $P_{F,(1)}R^{(1)} = 0$ modulo $(t_1, t_2, t_3)^2$. We omit to write down the matrix R^1 which contains already a lot of terms. The product $P_{F,1}R^1$ gets even worse, the first entry of this matrix reads as

$$\begin{aligned}
&- 12x_1x_3t_1t_2 - 4x_1x_2t_2^2 + 3x_3^2t_2^2 - 12x_3^2t_1t_3 - 10x_2x_3t_2t_3 + 3x_1x_2^2t_1t_2 + 9x_1^2x_3t_1t_2 - 3x_2x_3^2t_1t_2 \\
&+ 2x_1^2x_2t_2^2 - 5x_2^2x_3t_2^2 - 6x_1x_3^2t_2^2 - 6x_1^2x_2t_1t_3 + 6x_2^2x_3t_1t_3 + 2x_1^3t_2t_3 - 20x_1x_2x_3t_2t_3 - 8x_3^3t_2t_3 \\
&+ 4x_1^2x_3t_3^2 - 8x_2x_3^2t_3^2 - 9x_1^2x_2x_3t_1^2 - 18x_1^2x_2^2t_1t_2 + 9x_1^3x_3t_1t_2 + 6x_1x_2x_3^2t_1t_2 + 4x_1^3x_2t_2^2 \\
&+ 5x_1x_2^2x_3t_2^2 + 2x_1^2x_3^2t_2^2 - 5x_2x_3^3t_2^2 - 24x_1^3x_2t_1t_3 - 6x_1x_2^2x_3t_1t_3 + 3x_2x_3^3t_1t_3 + 2x_1^4t_2t_3 \\
&+ 6x_1^2x_2x_3t_2t_3 + 10x_2^2x_3^2t_2t_3 + 3x_1x_3^3t_2t_3 + 4x_1^3x_3t_3^2 + 2x_3^4t_3^2 + 9x_1^5t_1^2 + 9x_1^3x_2x_3t_1^2 \\
&+ 9/4x_1^3x_2^2t_1t_2 + 3x_1^4x_3t_1t_2 - 39/2x_1^2x_2x_3^2t_1t_2 + 3/4x_1^4x_2t_2^2 - 5x_1^2x_2^2x_3t_2^2 - 3/2x_1^3x_3^2t_2^2 \\
&+ 10x_1x_2x_3^3t_2^2 - 6x_1^4x_2t_1t_3 + 9/2x_1^2x_2^2x_3t_1t_3 - 9/4x_1x_2x_3^3t_1t_3 + 2x_1^5t_2t_3 + 3x_1^3x_2x_3t_2t_3 \\
&- 10x_1x_2^2x_3^2t_2t_3 - 1/4x_1^2x_3^3t_2t_3 + 7/2x_2x_3^4t_2t_3 + 4x_1^4x_3t_3^2 + 3x_1^2x_2x_3^2t_3^2 - x_1x_3^4t_3^2 \\
&- 27/4x_1^4x_2x_3t_1^2 - 27/16x_1^4x_2^2t_1t_2 - 9/4x_1^5x_3t_1t_2 + 147/8x_1^3x_2x_3^2t_1t_2 + 9/8x_1^5x_2t_2^2 \\
&+ 15/4x_1^3x_2^2x_3t_2^2 + 45/16x_1^4x_3^2t_2^2 - 15/4x_1^2x_2x_3^3t_2^2 - 15/4x_2^2x_3^4t_2^2 - 27/8x_1^3x_2^2x_3t_1t_3 \\
&- 45/16x_1^2x_2x_3^3t_1t_3 + 7/4x_1^4x_2x_3t_2t_3 + 15/2x_1^2x_2^2x_3^2t_2t_3 + 103/16x_1^3x_3^3t_2t_3 - 9x_1x_2x_3^4t_2t_3 \\
&- x_1^3x_2x_3^2t_3^2 - x_1^2x_3^4t_3^2 + 81/16x_1^5x_2x_3t_1^2 + 27/16x_1^6x_3t_1t_2 - 243/64x_1^4x_2x_3^2t_1t_2 \\
&- 45/4x_1^2x_2^2x_3^3t_1t_2 + 81/32x_1^5x_3^2t_2^2 + 6x_1^5x_3^2t_1t_3 - 531/64x_1^3x_2x_3^3t_1t_3 + 3/2x_1^5x_2x_3t_2t_3 \\
&+ 33/32x_1^4x_3^3t_2t_3 - 3/2x_1^2x_2x_3^4t_2t_3 + 3x_1^4x_2x_3^2t_3^2 - 3x_1^3x_3^4t_3^2 - 243/64x_1^6x_2x_3t_1^2 \\
&+ 81/32x_1^7x_3t_1t_2 - 9/2x_1^4x_2x_3^3t_1t_3.
\end{aligned}$$

However, in this example we are nearly finished: Modulo P_f this reduces to

$$\begin{pmatrix} -12t_1t_2\cdot x_2 & + & (7t_2^2-12t_1t_3)\cdot x_3^2 & + & 12t_2t_3\cdot x_1^3 \\ 16t_2^2\cdot x_2 & + & 16t_2t_3\cdot x_3^2 & + & 12t_1t_2\cdot x_1^3 \\ -16t_2t_3\cdot x_2 & - & 8t_3^2\cdot x_3^2 & + & (7t_2^2-12t_1t_3)\cdot x_1^3 \\ & & 0 & & \\ & & 0 & & \\ & & 0 & & \end{pmatrix}^t$$

and so $J_{(2)}$ is defined by $(t_1t_2, t_1t_3, t_2^2, t_2t_3, t_3^2)$. One checks that this not only gives a lift modulo $J_{(2)}+(t_1,t_2,t_3)^3$ but even modulo $J_{(2)}$, so the one-dimensional subgerm of $(\mathbb{C}^3,0)$ defined by $J_{(2)}$ forms the modular stratum of the $T_{4,3,3}$-singularity. A primary decomposition of $J_{(2)}$ is given as

$$J_{(2)} = (t_2,t_3)\cap\left[(t_1)+(t_2,t_3)^2\right],$$

so the modular stratum contains one smooth component corresponding to the one-parameter family $f+t_1x_1^3$, and, in addition, one embedded fat point.

3.2 On the implementation in Singular

We are now going to turn the discussion of the two previous sections into an explicit algorithmic description, quite close to the actual implementation in the SINGULAR-library `modular.lib` whose header is included as appendix A. For further technical details on the way the algorithm is coded we refer to the documented source code. Most of the steps in the algorithm written down below have a direct counterpart in SINGULAR and are self-explanatory, but some remarks are in order, concerning the normal form algorithm with respect to a local ordering.

The procedure `division(p,N,d,x)`: One of the crucial steps in the algorithm will be to reduce elements modulo P_f and write the remainder as a linear combination of the given $\mathbb{C}$-basis of $T^1(X_0)$. More generally, let N be a submodule of R^k, $R:=K[x_1,\ldots,x_n]$, and $n_1,\ldots,r_t$ a standard basis of N with respect to a local ordering, such that (R^k/N) has a finite K-basis $b_1,\ldots,b_t\in K[x]^k$ with d the maximal degree (in x) occurring. In addition, let $S:=R\otimes K[s_1,\ldots,s_e]$, and let $p\in S^k$. Here we choose a block ordering whose first block corresponds to the given local ordering on R.

Then the following procedure will give a standard representation of p with respect to $S\cdot N$ up to degree d:

1: **while** $p\neq 0$ **do**
2: $p := \mathtt{jet}(p,d,x)$
3: **if** there is i such that $Lt(n_i)|Lt(p)$ **then**
4: $p := p-\frac{Lt(p)}{Lt(n_i)}n_i$
5: $r_i := r_i+\frac{Lt(p)}{Lt(n_i)}$
6: **else**
7: $r := r+Lt(p)$
8: $p := p-Lt(p)$

Here we mean, for two terms $cx^\alpha e_i$ and $dx^\beta e_{i'}$ in $K[x]^k$, by $cx^\alpha e_i | dx^\beta e_{i'}$ that $i = i'$ and $\alpha_j \leq \beta_j$ for all j. In this case $\frac{cx^\beta e_i}{dx^\alpha e_i} := \frac{c}{d} x^{\beta-\alpha}$.

This procedure results in elements $r_1, \ldots, r_t \in S$ and $r \in S^k$ such that

$$p = r_1 n_1 + \ldots + r_t n_t + r \bmod (x)^{d+1} S^k,$$

and the remainder $r \in R^k$ is a $K[s]$-linear combination of the vector space basis $b_1, \ldots, b_t$ of the quotient. This 'normal form up to order d in x' can be seen as a way of forcing Buchberger's normal form algorithm to terminate in a local ordering by throwing away everything in oder $\geq d+1$ (i. e. taking the relative d-jet with respect to the variables $x_1, \ldots, x_n$).

This procedure had been suggested by B. Martin, and a first implementation into the SINGULAR-kernel had been done by the author for the purpose of applications involving local computations like the ones needed in the context we are dealing with here.

3.2.1 Computing the tangent space

Now, having explained how to work around this technical obstacle, we can describe what to do in SINGULAR in order to compute modular strata (or, more precisely, at least jets of it). First of all we explain how to compute the tangent space. To this end, we have to construct all necessary data out of the given input ideal $f = (f_1, \ldots, f_k)$. These data will then be used as well when computing higher order approximations of the modular stratum. Altogether it can be summarized as stated in the box below.

Lines 1–8 construct the preliminary data: A $\mathbb{C}$-basis of $T^1(X_0)$ is computed (respectively the algorithm is aborted in case X_0 is not an isolated singularity), and the matrices $P_{F,0}$ and R^0 are constructed. Lines 9-13 contain the main loop of the algorithm where the matrix associated to the bracket $T^0 \times T^1 \to T^1$ is computed. Afterwards, in lines 14–17 the resulting equations for the tangent space to the modular stratum are determined and we are done in case M_{X_0} is trivial. Otherwise, in lines 18–22 the rings corresponding to the restricted deformation are constructed and the lift R^1 of the relations is computed before returning the result. Experience shows that this last step (computing R^1) is the most expensive step of the algorithm.

Algorithm: `tangmodular` – computes the tangent space to the modular stratum

Input: $f = f_1, \ldots, f_k \in \mathbb{C}\{x_1, \ldots, x_n\}$ defining $(V(f), 0) = X_0$
Output: a list of two rings S_0, P_0 such that

- $S_0 = \mathbb{C}\{s_1, \ldots, s_e\}$, $P_0 = \mathbb{C}\{x_1, \ldots, x_n\} \otimes S_0$ and
- $F_1 = \sum_{i=1}^{e} s_i g_i \in P_0$ such that $f + F_1$ defines the maximal modular family up to order 2 in (s), $e = \dim(M_{X_0})$, together with an appropriate lift R^1 of the relations of P_f.
- In addition, $P_f = P_{F,0}$, $P_{F,1}$, R^0, R^1, τ, KB, m, d_{max} are exported.

1: Compute a $k \times \tau$-matrix KB containing a $\mathbb{C}$-basis of T^1 (using `sing.lib`)
2: $\tau := \texttt{ncols}(KB)$
3: $d_{max} := \max_{i,j}\{\deg KB_{i,j}\} + 1$
4: **if** $\tau = \infty$ **then**
5: exit *// not an isolated singularity!*
6: Construct the matrix $P_{F,0} = P_f$ defined as in (3.1.1)
7: $R^0 := \texttt{syz}(P_{F,0})$
8: $m := \texttt{ncols}(R^0)$
9: **for** $i = 1$ to τ **do**
10: **for** $j = 1$ to m **do**
11: $$tmp := \begin{pmatrix} KB_{1,i} & & \dots & KB_{k,i} & \frac{\partial KB_{1,i}}{\partial x_1} & \dots & \frac{\partial KB_{1,i}}{\partial x_n} \\ & \ddots & \dots & \ddots & \vdots & & \vdots \\ & KB_{1,i} & \dots & KB_{k,i} & \frac{\partial KB_{k,i}}{\partial x_1} & \dots & \frac{\partial KB_{k,i}}{\partial x_n} \end{pmatrix}$$
12: $tmp := \texttt{division}(tmp \cdot R^0_j, P_{F,0}, d_{max}, x)[2] = \sum_{l=1}^{\tau} c_l KB_l$
13: $bracket_{i,(j-1)\tau+1\dots j\tau} := (c_1, \dots, c_\tau)$
14: $S := \texttt{syz}(\texttt{transpose}(bracket))$
15: $e := \texttt{ncols}(S)$ *//embedding dimension of* M_{X_0}
16: **if** $S = 0$ **then**
17: `return`(0) *// trivial modular stratum* $\{0\}$
18: $S_0 := \mathbb{C}\{s_1, \dots, s_e\}$, $P_0 := \mathbb{C}\{x\} \otimes S_0$
19: $F_1 := \sum_{i=1}^{e} s_i \cdot (KB \cdot S)_i$
20: Construct the matrix $P_{F,1}$ out of F_1 as in (3.1.1)
21: $R^1 := \texttt{division}(P_{F,1} \cdot R^0, P_{F,0}, d_{max}, x)[1]$
22: `return`(S_0, P_0)

Remark 3.2.1. Obvious changes apply to the functor of deformations with section. Here we need a $\mathbb{C}$-basis of $T^1(X_0, \mathfrak{m}_{X_0})$ instead of a basis of $T^1(X_0)$ and we have to change the matrix P_f into

$$\begin{pmatrix} f_1 & & f_2 & & \dots & f_k & & x_1\frac{\partial f_1}{\partial x_1} & \dots & x_1\frac{\partial f_1}{\partial x_n} & \dots & x_n\frac{\partial f_1}{\partial x_1} & \dots & x_n\frac{\partial f_1}{\partial x_n} \\ & \ddots & & \ddots & \dots & & \ddots & \vdots & & \vdots & \dots & \vdots & & \vdots \\ & f_1 & & f_2 & \dots & & f_k & x_1\frac{\partial f_k}{\partial x_1} & \dots & x_1\frac{\partial f_k}{\partial x_n} & \dots & x_n\frac{\partial f_k}{\partial x_1} & \dots & x_n\frac{\partial f_k}{\partial x_n} \end{pmatrix}.$$

3.2.2 Higher order jets of the modular stratum

After performing the computations of algorithm `tangmodular` the situation is the following: We have constructed $F_1 \in \mathbb{C}\{x\} \otimes \mathbb{C}\{s_1, \dots, s_e\}$ such that $F_0 + F_1$ defines a modular family over $\mathbb{C}\{s_1, \dots, s_e\}/(s)^2$, $e = \text{e.dim}(M_{X_0})$, this gives a complete description of the tangent space to the modular stratum, and we have also found a matrix R^1 of lifts of the relations R^0 over $\mathbb{C}\{s, x\}/(s)^2$, i. e.

$$(P_{F,0} + P_{F,1}) \cdot (R^0 + R^1) = 0 \mod (s)^2.$$

For the explicit computation of the next-order approximation, we now use the interpretation of the obstructions to lifting modularity in Proposition 2.2.9 and Corollary 2.2.11 in terms

of flatness of the module $\tilde{T}^1(X/S)$, the quotient of $\mathcal{O}_S \otimes \mathbb{C}\{x\}^k$ by P_F. Thus we can now apply B. Martin's algorithm to compute flattening strata of modules over local rings, which we therefore want to recall now. We follow the description in [Mar02a, section 4], adapting it to our situation.

Theorem 3.2.2. *Let N be an $A := \mathcal{O}_S\{x\}$-module with presentation*

$$\mathcal{O}_S\{x\}^{k'} \to \mathcal{O}_S\{x\}^k \xrightarrow{P_F} N \to 0,$$

and suppose that N is finitely generated over $\mathcal{O}_S$. Let L^0 be a $\mathbb{C}$-basis of $N_0 := N \otimes_{\mathcal{O}_S} \mathcal{O}_S/\mathfrak{m}_S$ and R^0 a matrix with columns generating the kernel of $P_{F,0} := P_F \otimes_{\mathcal{O}_S} \mathcal{O}_S/\mathfrak{m}_S$. Then we can find R^i, L^i and J'_i, such that

$$P_{F,e}R^0 + \ldots + P_{F,0}R^e = J'_1L^{e-1} + J'_2L^{e-2} + \ldots + J'_eL^0 \tag{3.2.1}$$

holds in $A_e := A \otimes_{\mathcal{O}_S} \mathcal{O}_S/\mathfrak{m}_S^{e+1}$ for all e and the elements of the matrix $J'_1 + \ldots + J'_e$ define the e-jet $J_{(e)}$ of the flattening ideal of N.*

Here, as before, indices of homogeneous objects (with respect to the order in $\mathfrak{m}_S$) denote their order.

Proof. This is [Mar02a, Proposition 4.1]. We only recall the construction because we will use it for our description of the algorithm. Suppose one has constructed everything in A_{e-1}, hence we need to determine suitable matrices R^e, J'_e and L^{e-1} such that equation (3.2.1) holds in A_e. Now let

$$X := P_{F,e}R^0 + \ldots + P_{F,1}R^{e-1} - J'_2L^{e-2} - \ldots - J'_{e-1}L^1 \stackrel{!}{=} J'_1L^{e-1} + J'_eL^0 - P_{F,0}R^e,$$

so X is known from the step before as the left hand side of the equation. Reducing X modulo J'_1 yields some X' and we can find a lift L^{e-1} such that

$$X - X' = J'_1L^{e-1}.$$

Reducing X' further modulo $P_{F,0}$ produces a matrix X'', and we can lift

$$X' - X'' = P_{F,0}R^e$$

for a suitable matrix R^e. Finally, what remains can be represented in the $\mathbb{C}$-basis L^0 of N_0, i. e.

$$X'' = J'_eL^0$$

for some L^0. Putting everything together gives the desired equation (3.2.1) in A_e, and one checks that the objects constructed have the asserted properties. ∎

Remark 3.2.3.

(1) First of all, in our situation we may assume that $P_{F,k} = 0$ for all $k > 1$: Starting with $F_1 := f + \sum_{i=1}^{\tau} s_ib_i$, we may assume that this is the equation of the semi-universal deformation inside of which we are computing the modular stratum: This follows from the fact that lifting the syzygies of $f_1, \ldots, f_k$ is included in lifting the syzygies of P_f, so flatness of $\mathcal{O}_S\{x\}/(F)$ is preserved, i. e. the families under consideration are always deformations of $(V(f), 0)$.

*i. e. the minimal ideal J of A such that $N \otimes A/J$ is A/J-flat. See e. g. [Mar02a, section 2] for a proof of its existence.

(2) In the form stated above we could not apply this construction to our setting, where $N = \tilde{T}^1(X/S)$: In general, this is not a finitely generated $\mathcal{O}_S$-module. But nevertheless, this is only needed at the last step in the construction, where we want to lift the last reduction X'' with respect to a $\mathbb{C}$-basis of $N_0 = \mathbb{C}\{x\}^k/P_f$. But since this coincides with the obstruction element with respect to the small extension

$$0 \to (J_{(e)} + \mathfrak{m}_S^{e+1})/\mathfrak{m}_S(J_{(e)} + \mathfrak{m}_S^{e+1}) \to \mathcal{O}_S/\mathfrak{m}_S(J_{(e)} + \mathfrak{m}_S^{e+1}) \to \mathcal{O}_S/(J_{(e)} + \mathfrak{m}_S^{e+1}) \to 0,$$

X'' actually ends up in $T^1(X_0) \otimes \mathfrak{m}_S^{e+1}$, so we can use a finite $\mathbb{C}$-basis of $T^1(X_0)$ instead.

Translating the above construction into an algorithm we finally obtain:

Algorithm: `modular` – computes the modular stratum up to a given order

---

Input: $f = f_1, \ldots, f_k \in \mathbb{C}\{x_1, \ldots, x_n\}$ defining $(V(f), 0) = X_0$, and $d \geq 1$
Output: a list of two rings S_0, P_0 such that

- $S_0 = \mathbb{C}\{s_1, \ldots, s_e\}$, $P_0 = \mathbb{C}\{x_1, \ldots, x_n\} \otimes S_0$
- these rings contain $F = f + \sum_{i=1}^{e} s_i g_i \in P_0$ and ideals $J_{(1)}, \ldots, J_{(d)}$ such that $J_{(e)}$ defines the modular stratum of X_0 up to order e for all $e = 1, \ldots, d$.

1: $(S_0, P_0) :=$ `tangmodular`$(f_1, \ldots, f_k)$
   // so $P_{F,0}, P_{F,1}, R^0, R^1, \tau, KB, m, d_{max}$ are known
2: **if** $(S_0, P_0) = 0$ **then**
3: exit *// trivial modular stratum* $\{0\}$
4: $(R^2, tmp) :=$ `division`$(P_{F,1} \cdot R^1, P_{F,0}, d_{max}, x)$
5: Compute $J_2' \in Mat(\tau, m; S_0)$ with $tmp = KB \cdot J_2'$
6: $L^0 :=$ `kontrahom`$(\text{`ideal`}(KB), m)$ *// $\hat{=}$* Hom(`ideal`$(KB), \mathbb{C}\{x\}^m$)
7: $J_{(2)} :=$ `ideal`(J_2')
8: **for** $e = 3$ to d **do**
9: $X := P_{F,1} \cdot R^{e-1} - J_3' \cdot L^{e-3} - \ldots - J_{e-1}' \cdot L^1$
10: $X' :=$ `NF`(X, J_1')
11: $L^{e-1} :=$ `lift`$(X - X', J_1')$
12: $(R^e, X'') :=$ `division`$(X', P_{F,0}, d_{max}, x)$
13: Compute $J_e' \in Mat(\tau, m; S_0)$ with $X'' = KB \cdot J_e'$
14: $J_{(e)} :=$ `ideal`$(J_2' + \ldots + J_e')$
15: `return`(S_0, P_0)

In the first three lines, the algorithm calls `tangmodular` to obtain the tangent space and the data computed there. It finishes in case the result is trivial. The next lines 4–7 contain the computation in order two, and the remaining lines 8–14 are the main loop, calculating the data of the next order up to the given bound d, which is done exactly as described in the proof of Theorem 3.2.2.

Remark 3.2.4. This part of the computation of M_{X_0} does not require any modifications in order to adjust it to the functor of deformations with section: All differences concerning the objects involved are already taken into account in `tangmodular`, where the necessary modifications to $P_{F,0}, P_{F,1}$ and KB are made.

Remark 3.2.5. In general, the algorithm presented above will only return e-jets of the modular stratum M_{X_0}, and only in lucky cases we obtain polynomial equations for M_{X_0}, i. e. $J_{(e)}$ defines M_{X_0} for some e. Nevertheless, one can perform some tests which, when positive, imply that one can stop and has obtained the correct equations:

(1) If $(s)^{e+1} \subseteq J_{(e)}$ in some iteration, then we will not get anything new in further degree, in particular the modular stratum is Artinian in this case.

(2) Check whether the algorithm has found a lift modulo $J_{(e)}$, i. e. whether $P_{F,(1)} \cdot R^{(e)}$ vanishes modulo $J_{(e)}$. This test usually takes longer and implies that the algorithm has found a set of polynomial generators for the ideal of M_{X_0} which of course does not have to be the case.

For practical usage, it seems more convenient to let the algorithm run for some loops (in fact, for more complicated input, it will not terminate in higher degree since data becomes much to memory-consuming), and to perform test (2) afterwards on the data returned.

Let us give some examples to demonstrate the functionality of our implementation in the SINGULAR-library `modular.lib`. A more technical description of the procedures included is contained in appendix A.

Example 3.2.6. As a simple example, take the plane curve $f := x^6 + y^8 + x^4y^6 \in \mathbb{C}\{x, y\}$.

(1) We compute its modular stratum and a maximal modular family using the command `modular` provided by `modular.lib` ('`==>`' denotes SINGULAR output):

```
LIB "modular.lib";
ring r=0,(x,y),ds;
poly f=x6+y8+x4y6;
list L=modular(f,4);

==> // Equation(s) of modular family:
==> f1[1,1]=x3y4C+x4y5B+x3y6A
==>
==> // Embedding dimension: 3
==> // ** not all weights are positive!
==>
==> // Ideal of modular stratum up to order 4:
==> J(4)[1]=A2
==> J(4)[2]=AB
==> J(4)[3]=B2
==>
==> // modular has computed a list, say L, of two rings:
==> //   - basering of the modular stratum, containing J(k), the k-jet
==> //     of the ideal defining the modular stratum for k<=4;
==> //   - basering of embedding space of the total deformation space.
==> // to make them accessible type def So=L[1]; def Po=L[2];
```

Hence, the maximal modular family is given by $f + A \cdot x^3y^6 + B \cdot x^4y^5 + C \cdot x^3y^4$ over the base $\mathbb{C}\{A, B, C\}/(A, B)^2$ (the computation is only done up to order 4, but now one can check that all relations not only lift modulo $J(4) + (A, B, C)^5$ but already modulo $J(4)$). The returned list L contains two rings S_0 and P_0 as stated in the formulation of the algorithm, in which the respective objects R^i, L^i etc. are defined.

(2) For computations with respect to $Def^s_{X_0}$, one has to pass an additional non-zero third parameter to SINGULAR when invoking `modular` resp. `tangmodular`, e. g. for the same singularity as in (1):

```
L=modular(f,4,1);

==> // Equation(s) of modular family:
==> f1[1,1]=x5E+x3y4D+y7A+x4y5C+x3y6B
==>
==> // Embedding dimension: 5
==>
==> // Ideal of modular stratum up to order 4:
==> J(4)[1]=A2
==> J(4)[2]=AB-17/18AE
==> J(4)[3]=B2-3/2BE
==> J(4)[4]=AC
==> J(4)[5]=BC-5/84AE-4/21CE
==> J(4)[6]=C2
==> J(4)[7]=AD
==> J(4)[8]=AE-14/13CE
==> J(4)[9]=BE+4/9E2
==> J(4)[10]=DE
==> J(4)[11]=E2
==>
==> // modular has computed a list, say L, of two rings:
==> //   - basering of the modular stratum, containing J(k), the k-jet
==> //     of the ideal defining the modular stratum for k<=4;
==> //   - basering of embedding space of the total deformation space.
==> // to make them accessible type def So=L[1]; def Po=L[2];
```

Again, one can check that this already gives a complete description of the modular stratum, and a primary decomposition of its ideal is

$$\big[(A,E)+(B,C)^2\big]\cap \\ (D,E^2,BE,13AE-14CE,C^2,546BC-139CE,AC,B^2,117AB-119CE,A^2),$$

where the family over the first component coincides with the modular stratum for Def_{X_0}, and the second ideal describes a non-reduced point in the hyperplane $D=0$.

If one is only interested in the embedding dimension and the equations of the modular family (or in checking whether the modular stratum is trivial), one can simply use the command `tangmodular`:

Example 3.2.7. For instance, the following computation shows that the cone over the rational normal curve of degree 4 has trivial modular stratum:

```
ring r=0,(x(0..4)),ds;
matrix M[2][4]=x(0),x(1),x(2),x(3),
               x(1),x(2),x(3),x(4);
ideal I=minor(M,2);
print(I);
==> -x(3)^2+x(2)*x(4), -x(2)*x(3)+x(1)*x(4), -x(1)*x(3)+x(0)*x(4),
```

```
==>  x(2)^2-x(1)*x(3),  x(1)*x(2)-x(0)*x(3), -x(1)^2+x(0)*x(2)

tangmodular(I);
==> // trivial modular stratum
==> 0
```

All previous examples could be computed quite fast, none of the computations takes more than a few seconds to finish. In contrast, consider the following singularity:

Example 3.2.8. Let $I \subseteq \mathbb{C}\{x_0, \ldots, x_5\}$ be given as the ideal of all but the neighboring 2×2-minors of the matrix of indeterminates

$$M := \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_0 \end{pmatrix}.$$

The computation of the embedding dimension of the modular stratum is performed quite fast (approx. 15 seconds), but then it takes about 60 more seconds to compute lifts of the syzygies in R^0. The result is (with somewhat shortened output):

```
ring r=0,(x(0..5)),ds;
ideal I=x(0)*x(3)-x(1)*x(2), x(1)*x(4)-x(2)*x(3), x(2)*x(5)-x(3)*x(4),
        x(3)*x(0)-x(4)*x(5), x(0)*x(4)-x(1)*x(3), x(1)*x(5)-x(2)*x(4),
        x(2)*x(0)-x(3)*x(5), x(0)*x(5)-x(1)*x(4), x(1)*x(0)-x(2)*x(5);

tangmodular(I);

==> // Equation(s) of modular family:
==> [...]

==> // Embedding dimension: 25
==> [...]
```

Hence, in this example the tangent space to the modular stratum coincides with the tangent space of S, since 25 equals the Tjurina number of $(V(I), 0)$. But now we are not able to compute higher jets of the modular stratum: The next step would consist of computing the product $P_{F,1} \cdot R^1$ and reducing it modulo $P_{F,0}$. Here we reach practical limits: Because of the high embedding dimension of the modular stratum we cannot reduce the number of variables, in addition R^0 contains as much as 389 columns, and even printing the resulting 9×389-matrix $P_{F,1} \cdot R^1$ produces more than 50 MBytes (!) of output.

Nevertheless, this is an interesting example because we obtain that, in contrast to Example 3.2.7 where the modular stratum has been trivial, for this singularity its embedding dimension is as big as possible.

We will come back to these two examples in section 4.3 where we discuss some general facts concerning the modular stratum of quasihomogeneous singularities.

Chapter 4

Examples of modular strata

In this final part of our work we are going to apply the methods and the algorithm developed so far to the study of some concrete examples and classes of singularities. The theoretical criteria on a germ to be modular usually do not enable us to compute the modular stratum for a general given type of singularities, since this would require precise information on the module of derivations which, in general, one cannot get. Thus the present implementation can be useful to compute concrete examples, ant it has it possible to exhibit phenomena which have not been known before.

As a first example, we give an answer to the question which had formed the motivation of our study of deformations with section and which had been formulated in the introduction of this thesis: Is a splitting of the singular locus along a modular family with respect to the functor $Def^s_{X_0}$ possible? The answer is yes – and the first example below will illustrate this phenomenon. Thus, in the following we will concentrate mainly on the 'usual' deformation functor.

As an example of a whole class of singularities where we can give a complete description of the modular stratum, we determine the modular strata of all hypersurface singularities of modality zero and one. Having established this result we compare it to the adjacency table of this class of singularities.

In the last two sections we take the characterization of the modular stratum of quasihomogeneous complete intersections (of positive dimension) in chapter two as a starting point: We investigate what happens when these assumptions are no longer fulfilled – i. e. what one may encounter in dimension zero or for non-complete intersections: The Euler relation still imposes a restriction on the dimension of the modular stratum, but in general the modular stratum may become much more complicated.

Finally we restrict ourselves to hypersurfaces and try to give some idea of what kind of modular strata may occur when passing to semi-quasihomogeneous singularities.

4.1 Splitting of the singular locus

To start with this program, we study the following example below. It illustrates how the isolated singular point of the original singularity can break up along a modular family, demonstrating that τ-constantness along the modular stratum is only satisfied by considering the fibres as multigerms. Intuitively one might have expected that this phenomenon cannot occur when studying deformations having a section – but in fact, this is possible in that case, too.

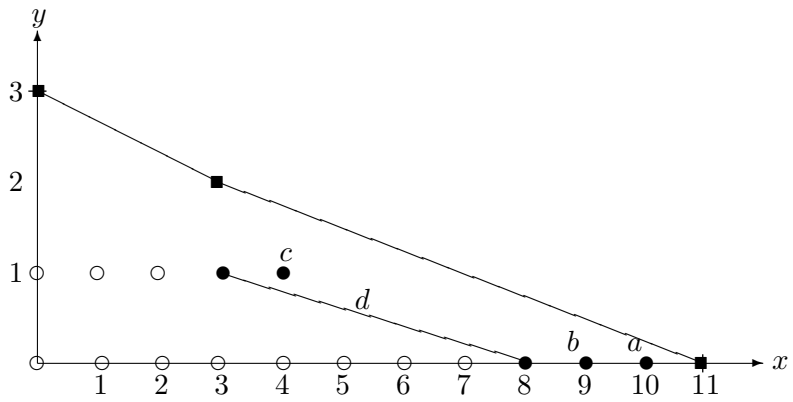
Example 4.1.1. Consider the plane curve singularity X_0 defined by $f := y^3 + x^3y^2 + x^{11} \in \mathbb{C}\{x, y\}$, with Tjurina number $\tau = 16$. The computation of its modular stratum M_{X_0} in SINGULAR is carried out easily, and the result is: M_{X_0} has embedding dimension 4, and a primary decomposition of its defining ideal is given as

$$(a, c, d) \cap (a^2 - 4b, c, d) \cap (d^2, cd, c^2, bd, b^2, abc, 112bc + 109ad, 109a^2c - 100bc, a^3 - 4ab + 38ac)$$

in $\mathbb{C}\{a, b, c, d\}$. Hence M_{X_0} decomposes into two smooth curves C_1 and C_2, and one embedded point. The corresponding modular family is given by

$$F_M = f + a \cdot x^{10} + b \cdot x^9 + c \cdot x^4y + d \cdot (x^3y - \tfrac{11}{2}x^8),$$

and can be illustrated as follows:



Thus we obtain two one-parameter τ-constant families with special fibre X_0: One is defined by $f_t := f + tx^9$ over the cure C_1. It has, for all t, 0 as only singular point, its Tjurina number is 16, and it becomes quasihomogeneous for $t \neq 0$. On the other hand, consider the family defined by $\tilde{f}_t := f + 2tx^{10} + t^2x^9$ over C_2. Then, for $t \neq 0$, $\tilde{X}_t = V(f_t)$ has two singular points, namely $(0, 0)$ and $(-t, 0)$ with Tjurina numbers 15 and 1, respectively. I. e., the singularity at 0 becomes slightly 'simpler' in the fibers outside the origin, with an additional A_1-singularity popping up. This illustrates that τ-constancy really means constancy of the Tjurina number of the *multi-germ* $\tilde{X}_t$ (in accordance with the principle of conservation of number in analytic geometry, cf. [dJP00, Chapter 6]).

The same splitting also occurs along the $Def^s_{X_0}$-modular stratum: Its embedding dimension increases by one, with (similar) maximal modular family

$$F_M = f + a \cdot x^{10} + b \cdot x^9 + c \cdot x^5y + d \cdot x^4y + e \cdot (y^2 + \tfrac{3}{5}x^3y + \tfrac{11}{30}x^8)$$

over $\mathbb{C}\{a, b, c, d, e\}$. The modular stratum again splits into two smooth curves $V(a, c, d, e)$ and $V(a^2 - 4b, c, d, e)$ defining the same one-parameter subfamilies, and one embedded point (different from the one above, of course). In particular, we again have the modular family $f_t := f + 2tx^{10} + t^2x^9$ where the singular locus splits for $t \neq 0$.

A similar example exhibiting this phenomenon already appeared in [Mar02a]. In fact, this example had been the starting point for investigating what happens to the modular stratum when considering the functor of deformations with section instead of the 'ordinary' deformation functor.

4.2 Hypersurfaces of modality 0 and 1

A complete classification of these classes of singularities is well-known, cf. [AGZV85], for example, and we follow the notations used there. Apart from the $A-D-E-$singularities, which clearly have trivial modular stratum $\{0\}$ (they are simple!), there is only one infinite series of singularities of modality ≤ 1: The $T_{p,q,r}$-series. Therefore, first of all we show that all but finitely many singularities of this series have Artinian modular stratum, and we give a description in fairly concrete terms*:

Proposition 4.2.1. *Let $p_1 \geq p_2 \geq p_3 \in \mathbb{N}$ such that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} \leq 1$. Then the modular stratum of the singularity $T_{p_1,p_2,p_3} := (V(f_p),0) \subseteq (\mathbb{C}^3,0)$ given by*

$$f_p := \frac{x_1^{p_1}}{p_1}+\frac{x_2^{p_2}}{p_2}+\frac{x_3^{p_3}}{p_3}+\lambda\cdot x_1x_2x_3, \qquad \lambda\in\mathbb{C},\ ^{\dagger}$$

is shown in the following table:

p_1	p_2	p_3	F_M	J_M	e.dim	dim
6	3	2	$f_p+s_1x_1x_2x_3$	(0)	1	1
4	4	2	$f_p+s_1x_1x_2x_3$	(0)	1	1
3	3	3	$f_p+s_1x_1x_2x_3$	(0)	1	1
7	3	2	$f_p+s_1x_1^6+s_2(x_1^5-\frac{1}{84}x_1^2x_2)$	$(s_2)\cap(s_1,s_2^2)$	2	1
6	4	2	$f_p+s_1x_1^5+s_2x_2^3$	$(s_1)\cap(s_1^2,s_2)$	2	1
6	3	3	$f_p+s_1x_1^5+s_2x_2^2+s_3x_3^2$	$(s_1,s_2)\cap(s_1,s_3)\cap(s_1^2,s_2,s_3)$	3	1
5	4	2	$f_p+s_1x_1^4+s_2x_2^3$	$(s_2)\cap(s_1,s_2^2)$	2	1
4	4	3	$f_p+s_1x_1^3+s_2x_2^3+x_3^2$	$(s_1,s_2)\cap[(s_3)+(s_1,s_2)^2]$	3	1
4	3	3	$f_p+s_1x_1^3+s_2x_2^2+s_3x_3^2$	$(s_2,s_3)\cap[(s_1)+(s_2,s_3)^2]$	3	1
other	3	2	$f_p+s_1x_1^{p_1-1}+s_2(x_1^{p_1-2}-\frac{1}{12p_1}x_1^2x_2)$	$(s_1,s_2)^2$	2	0
other	≥ 4	2	$f_p+s_1x_1^{p_1-1}+s_2x_2^{p_2-1}$	$(s_1,s_2)^2$	2	0
other	*other*	≥ 3	$f_p+s_1x_1^{p_1-1}+s_2x_2^{p_2-1}+s_3x_3^{p_3-1}$	$(s_1,s_2,s_3)^2$	3	0

Here F_M is the equation of the maximal modular family and J_M the ideal in $\mathbb{C}\{s_1,s_2,s_3\}$ (resp. $\mathbb{C}\{s_1,s_2\}$ in case $p_3=2$) defining the modular stratum. The last columns indicate the embedding dimension resp. the dimension of the modular stratum.

Proof. The first two parts of the table follow from a direct computation in SINGULAR (note that its first part is also an application of Proposition 2.4.2). For the proof that all others have Artinian modular stratum we can proceed as follows: Let us show this for the case that $p_1,p_2,p_3 \geq 4$ (the remaining cases $p_3=2,3$ can be treated similarly, one has to make some modifications for the derivations and lifts).

We simply write $f=f_p$. Its Milnor number $\mu(f)$ equals $p_1+p_2+p_3-1$, and a $\mathbb{C}$-basis of its local algebra $Q(f)=\mathbb{C}\{x_1,x_2,x_3\}/J(f)$ is given by

$$B_\mu := \{1,x_1,\ldots,x_1^{p_1-1},x_2,\ldots,x_2^{p_2-1},x_3,\ldots,x_3^{p_3-1},x_1x_2x_3\}.$$

*We should note that the below equations do not describe the full modular stratum of the corresponding singularities – cf. the footnote $(*)$ on p. 53.

†under the restrictions $\lambda^3+27\neq 0$ for $(3,3,3)$, $\lambda^2\neq 4$ for $(4,4,2)$, $4\lambda^3+27\neq 0$ for $(6,3,2)$ and $\lambda\neq 0$ otherwise.

For the Tjurina number τ we have $\tau = \mu - 1$, and we obtain a $\mathbb{C}$-basis of T^1 as $B_\tau := B_\mu \setminus \{x_1x_2x_3\}$. In particular, the kernel of the multiplication map $Q(f) \xrightarrow{\cdot f} Q(f)$ has dimension $\mu - 1$, hence equals $(x_1, x_2, x_3)/J(f)$. Since this is isomorphic to $T^0_\bullet$ by Proposition 2.3.2, we conclude that it suffices to lift three derivations $\delta_{x_1}, \delta_{x_2}, \delta_{x_3} \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}\{x_1, x_2, x_3\})$ such that $\delta_{x_i}(f) = x_i f$. Explicitly we can take (cf. [Pal85]):

$$\delta_{x_1} := \left(\frac{x_1^2}{p_1} + u \cdot x_2^{p_2-2}x_3^{p_3-2}\right)\frac{\partial}{\partial x_1} + \left(\frac{x_1x_2}{p_2} - u \cdot x_3^{p_3-1}\right)\frac{\partial}{\partial x_2} + \left(\frac{x_1x_3}{p_3} + u \cdot x_1x_3\right)\frac{\partial}{\partial x_3},$$

where $u := \left(1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3}\right)\left(1 + x_1^{p_1-3}x_2^{p_2-3}x_3^{p_3-3}\right)^{-1}$ is a unit in $\mathbb{C}\{x_1, x_2, x_3\}$, and by interchanging the variables cyclically, we also find $\delta_{x_2}, \delta_{x_3}$. Now compute the Lie bracket $[-,-] : T^0_\bullet \times T^1 \to T^1$. We get (for $j = 1, 2, 3$ and $i = 1, \ldots, p_j - 1$):

- $[\delta_{x_j}, x_j^i] = (\frac{i}{p_j} - 1)x_j^{i+1}$ modulo $(f, J(f))$, hence this goes to zero in T^1 if and only if $i = p_j - 1$.
- $[\delta_{x_j}, x_{j'}^i] = 0$ for all i and $j \neq j'$.

Thus we obtain the following modular family over $\mathbb{C}\{s_1, s_2, s_3\}/(s)^2$: $F := f + f_1$ where $f_1 := \sum_{j=1}^3 s_j x_j^{p_j-1}$. Now we have to compute lifts of the derivations δ_{x_j} modulo $(s)^2$, here again it suffices to consider δ_{x_1}. Thus we want to find $\tilde{\delta} = \sum_{i,j=1}^3 s_j g_{i,j}(x)\frac{\partial}{\partial x_i}$ and $\tilde{h} = s_1h_1(x) + s_2h_2(x) + s_3h_3(x)$ such that

$$(\delta_{x_1} + \tilde{\delta})(f + f_1) = (x_1 + \tilde{h}) \cdot (f + f_1) \mod (s)^2,$$

which finally yields the equations

$$\begin{aligned}
s_1 \cdot \left(g_{1,1}\tfrac{\partial f}{\partial x_1} + g_{2,1}\tfrac{\partial f}{\partial x_2} + g_{3,1}\tfrac{\partial f}{\partial x_3} - h_1 \cdot f\right) &= s_1 \cdot \left(\tfrac{1}{p_1}x_1^{p_1} - u \cdot x_1^{p_1-2}x_2^{p_2-2}x_3^{p_3-2}\right)\\
s_2 \cdot \left(g_{1,2}\tfrac{\partial f}{\partial x_1} + g_{2,2}\tfrac{\partial f}{\partial x_2} + g_{3,2}\tfrac{\partial f}{\partial x_3} - h_2 \cdot f\right) &= s_2 \cdot \left(\tfrac{1}{p_2}x_1x_2^{p_2-1} + u \cdot x_2^{p_2-2}x_3^{p_3-1}\right)\\
s_3 \cdot \left(g_{1,3}\tfrac{\partial f}{\partial x_1} + g_{2,3}\tfrac{\partial f}{\partial x_2} + g_{3,3}\tfrac{\partial f}{\partial x_3} - h_3 \cdot f\right) &= s_3 \cdot \left(\tfrac{1}{p_3}x_1x_3^{p_3-1} - u \cdot x_1x_3^{p_3-1}\right).
\end{aligned}$$

A solution to the first of these equations is given as

$$\begin{aligned}
g_{1,1} &= ux_1 \cdot (a_1 + b_1x_1^{p_1-3}x_2^{p_2-3}x_3^{p_3-3} + c_1x_1^{p_1-4}x_2^{2p_2-4}x_3^{p_3-4})\\
g_{2,1} &= ux_2 \cdot (a_2 + b_2x_1^{p_1-3}x_2^{p_2-3}x_3^{p_3-3} + c_2x_1^{p_1-4}x_2^{2p_2-4}x_3^{p_3-4} + x_1^{p_1-4}x_2^{p_2-4}x_3^{2p_3-4})\\
g_{3,1} &= ux_3 \cdot (a_3 + b_3x_1^{p_1-3}x_2^{p_2-3}x_3^{p_3-3} + c_3x_1^{p_1-4}x_2^{2p_2-4}x_3^{p_3-4})\\
h_1 &= u \cdot (a_0 + b_0x_1^{p_1-3}x_2^{p_2-3}x_3^{p_3-3} + c_0x_1^{p_1-4}x_2^{2p_2-4}x_3^{p_3-4}),
\end{aligned}$$

where

$$\begin{array}{llll}
a_0 = & b_0 = -\dfrac{1}{p_1 \cdot (1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3})} & c_0 = \dfrac{1}{1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3}} \\
a_1 = & b_1 = \dfrac{1}{p_1}\left(\dfrac{1}{1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3}} - a_0\right) & c_1 = -\dfrac{c_0}{p_1} \\
a_2 = & b_2 = -\dfrac{a_0}{p_2} & c_2 = -\dfrac{c_0}{p_2} \\
a_3 = -\dfrac{a_0}{p_3} & b_3 = -1 - a_0 - a_1 - a_2 & c_3 = c_0 \cdot \left(\dfrac{1}{p_1} + \dfrac{1}{p_2} - 1\right).
\end{array}$$

A similar calculation gives $g_{i,2}, g_{i,3}, h_2$ and h_3. We now look at the obstructions to finding $\hat{\delta}$ and $\hat{h}$ (homogeneous of degree 2 in s) such that

$$(\delta_{x_1} + \tilde{\delta} + \hat{\delta})(f + f_1) = (x_1 + \tilde{h} + \hat{h})(f + f_1) \mod (s)^3,$$

so, looking at degree 2, we would have to solve

$$\tilde{\delta}(f_1) - \tilde{h}f_1 = \hat{h}f - \hat{\delta}(f).$$

Sorting the left hand side with respect to coefficients of $s_1 s_j$, one obtains as coefficient of s_1^2:

$$g_{1,1}\frac{\partial x_1^{p_1-1}}{\partial x_1} - h_1 x_1^{p_1-1} = u(a_1(p_1-1) - a_0)x_1^{p_1-1} + r,$$

where r vanishes modulo $(f, J(f))$, but the coefficient of $x_1^{p_1-1}$ is non-zero. An analogous computation for the other monomials $s_1 s_j$ shows that it is not possible to lift the derivation further, and by considering the other two derivations δ_{x_2} and δ_{x_3} as well, we see that we cannot lift them all simultaneously further than modulo $(s_1, s_2, s_3)^2$, so this defines a maximal modular subgerm. ∎

Having determined the modular strata of the T_{p_1,p_2,p_3}-series singularities, we can complete the table of modular strata for singularities of modality 0 and 1 (according to Arnold's classification list [AGZV85]) as follows:

Proposition 4.2.2. *The following tables show the modular strata of all zero- and unimodal hypersurface singularities, except for the T-series singularities (including the parabolic singularities $P_8 \hat{=} T_{3,3,3}$, $X_9 \hat{=} T_{4,4,2}$ and $J_{10} \hat{=} T_{6,3,2}$), for which the modular strata are listed in Proposition 4.2.1:*

- *Zeromodal singularities:*

A_k	D_k	E_6	E_7	E_8
x^{k+1}	$x^2y + y^{k-1}$	$x^3 + y^4$	$x^3 + xy^3$	$x^3 + y^5$

  All these singularities have trivial modular stratum.

- *Exceptional unimodal singularities:*

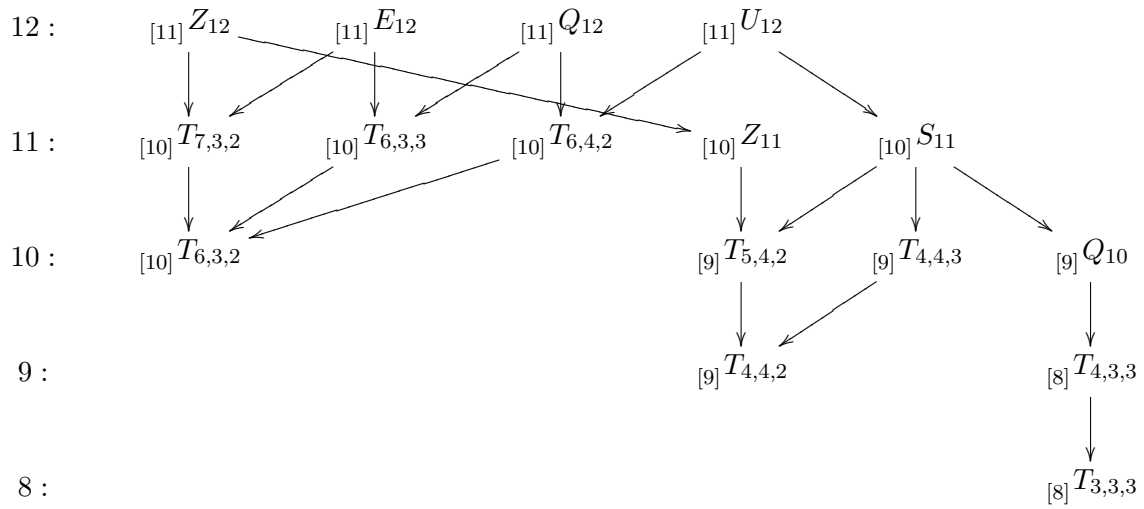
Type	f	F_M	J_M
E_{12}	$x^3 + y^7 + \lambda xy^5$	$f + s_1xy^4 + s_2(xy^3 + \frac{7}{5\lambda}y^5)$	$J_M = (s_1, s_2)^2$
E_{13}	$x^3 + xy^5 + \lambda y^8$	$f + s_1y^7 + s_2(xy^3 + \frac{8}{5\lambda}y^6)$	$J_M = (s_1, s_2)^2$
E_{14}	$x^3 + y^8 + \lambda xy^6$	$f + s_1xy^5 + s_2(xy^4 + \frac{4}{3\lambda}y^6)$	$J_M = (s_1, s_2)^2$
Z_{11}	$x^3y + y^5 + \lambda xy^4$	$f + s_1xy^3 + s_2y^4$	$J_M = (s_1, s_2)^2$
Z_{12}	$x^3y + xy^4 + \lambda x^2y^3$	$f + s_1xy^3 + s_2x^2y^2$	$J_M = (s_1, s_2)^2$
Z_{13}	$x^3y + y^6 + \lambda xy^5$	$f + s_1xy^4 + s_2y^5$	$J_M = (s_1, s_2)^2$
W_{12}	$x^4 + y^5 + \lambda x^2y^3$	$f + s_1xy^3 + s_2x^2y^2$	$J_M = (s_1, s_2)^2$
W_{13}	$x^4 + xy^4 + \lambda y^6$	$f + s_1x^2y^2 + s_2y^5$	$J_M = (s_1, s_2)^2$
Q_{10}	$x^3 + y^4 + yz^2 + \lambda xy^3$	$f + s_1y^3 + s_2xy^2 + s_3xz$	$J_M = (s_1, s_2, s_3)^2$
Q_{11}	$x^3 + y^2z + xz^3 + \lambda z^5$	$f + s_1xy + s_2xz^2 + s_3z^4$	$J_M = (s_1, s_2, s_3)^2$
Q_{12}	$x^3 + y^5 + yz^2 + \lambda xy^4$	$f + s_1y^4 + s_2xy^3 + s_3xz$	$J_M = (s_1, s_2, s_3)^2$
S_{11}	$x^4 + y^2z + xz^2 + \lambda x^3z$	$f + s_1x^2z + s_2x^2y + s_3x^3$	$J_M = (s_1, s_2, s_3)^2$
S_{12}	$x^2y + y^2z + xz^3 + \lambda z^5$	$f + s_1xz + s_2yz^2 + s_3z^4$	$J_M = (s_1, s_2, s_3)^2$
U_{12}	$x^3 + y^3 + z^4 + \lambda xyz^2$	$f + s_1yz^2 + s_2xz^2 + s_3xyz$	$J_M = (s_1, s_2, s_3)^2$

The families shown above are valid for nonzero values of the modulus parameter λ (which are all semi-quasihomogeneous with $\mu - \tau = 1$). In each case, the modular stratum is Artinian, with embedding dimension equal to the embedding dimension of the respective singularity.

For $\lambda = 0$ the resulting quasihomogeneous singularities have trivial modular stratum.

Proof. For the zeromodal singularities it is obvious that their modular stratum is trivial (besides, the result also follows directly from Proposition 2.4.2), and for the remaining unimodal ones the modular strata have been computed using SINGULAR. ∎

We would like to discuss now how the positive-dimensional modular strata popping up at the beginning of the T-series fit into the picture of adjacencies of unimodal hypersurface singularities. The following picture shows the beginning of their adjacency diagram (see [Bri79] for a complete list of all adjacencies). The first number in each row is the Milnor number of the singularities in that row, whereas the subscript to the left of each type stands for its Tjurina number:



In addition, there are several additional adjacencies between the exceptional and other singularities of the T-series, as well as the obvious adjacencies $T_{p,q,r} \to T_{p',q',r'}$ for $p \geq p'$, $q \geq q'$, $r \geq r'$ which we have not included because they are not important for the present discussion. Now there are two types of one-dimensional modular strata in the list of T-series singularities:

(1) For $T_{6,3,2}$, $T_{4,4,2}$ and $T_{3,3,3}$ we obtain the one-dimensional family of non-isomorphic singularities of the given type.

(2) The other six triples (p, q, r) giving one-dimensional modular strata are precisely those that are directly adjacent to the three previous ones, as shown in the diagram above. The adjacencies

$$\begin{array}{lll} T_{7,3,2} \to T_{6,3,2} & T_{6,4,2} \to T_{6,3,2} & T_{6,3,3} \to T_{6,3,2} \\ T_{5,4,2} \to T_{4,4,2} & T_{4,4,3} \to T_{4,4,2} & T_{4,3,3} \to T_{3,3,3} \end{array}$$

coincide with those adjacencies where the Milnor number decreases by one whereas the Tjurina number stays constant – and the one-dimensional smooth subfamily of the maximal modular deformation induces the respective adjacency.

All other adjacencies between unimodal singularities do not leave the Tjurina number fixed, therefore the modular strata of $T_{p,q,r}$ reflect them only infinitesimally.

Remark 4.2.3. In that context, we should note that A. G. Alexandrov has obtained a list of normal forms of one-dimensional quasihomogeneous complete intersections in [Ale82], which in particular includes the singularities $T_{6,3,2}$ and $T_{4,4,2}$. By Proposition 2.4.2 such singularities give rise to examples of singularities with smooth maximal modular families. In [Ale00] he has shown how one can construct a moduli space of curves which are unimodal quasihomogeneous complete intersections and have Milnor number less than 17 by gluing together appropriate modular families. Moreover, it is shown there that this moduli space admits a covering by either the complex plane or the punctured complex plane.

4.3 Quasihomogeneous complete intersections

The quasihomogeneous singularities in the table of Proposition 4.2.2 (i. e., take $\lambda = 0$) are examples of the assertion in Proposition 2.4.2, which contains a complete description of the modular stratum (with respect to both deformation functors) of quasihomogeneous complete intersections in positive dimension. Let us recall its formulation from section 2.4:

Proposition 2.4.2. *The modular stratum with respect to the functors Def_{X_0} and $Def^s_{X_0}$ of a quasihomogeneous isolated complete intersection singularity X_0 of positive dimension is reduced and smooth.*

Its dimension equals the dimension of the eigenspace (with respect to the eigenvalue zero) of the linear operator $[\delta_E, -] : T^1(X_0) \to T^1(X_0)$ – resp. $T^1(X_0, \mathfrak{m}_{X_0}) \to T^1(X_0, \mathfrak{m}_{X_0})$ – defined by the Euler derivation δ_E.

We want to show that in this statements all assumptions made on X_0 are necessary by means of some examples, and to explain what one can still say for arbitrary quasihomogeneous singularities.

First of all, recall that the main ingredient in its proof had been Aleksandrov's description of $T^0(X_0) \simeq T^0(X_0, \mathfrak{m}_{X_0})$ for such singularities in [Ale85]: It can be generated by the Euler derivation δ_E, together with the trivial ones, and we can lift the Euler derivation precisely along the subgerm of the base space corresponding to the kernel of the map $T^1(X_0) \to T^1(X_0)$, $g \mapsto [\delta_E, g]$. If X_0 is not a complete intersection or has dimension zero, then there may exist more non-trivial derivations that have to be lifted along a modular family, but, anyway, the Euler derivation is still present, and it restricts the modular stratum to the subspace described above. This argument proves:

Proposition 4.3.1. *Let X_0 be an isolated quasihomogeneous singularity. Then the embedding dimension (and thus the dimension) of its modular stratum is bounded above by the dimension of the eigenspace of zero of the endomorphism $[\delta_E, -] \in \mathrm{End}_{\mathbb{C}}(T^1(X_0))$ (resp. $\mathrm{End}_{\mathbb{C}}(T^1(X_0, \mathfrak{m}_{X_0}))$ when considering deformations with section) defined by the Euler derivation δ_E.*

Remark 4.3.2. Proposition 4.3.1 also gives a sufficient condition on a quasihomogeneous germ X_0 to have trivial modular stratum: If 0 is not an eigenvalue of the linear operator $[\delta_E, -]$ defined on $T^1(X_0)$ resp. $T^1(X_0, \mathfrak{m}_{X_0})$, then this germ must have trivial modular stratum.

This seems the best what one can get for arbitrary quasihomogeneous singularities. The examples computed at the end of chapter three demonstrate the two possible extreme cases for $\dim_{\mathbb{C}} T^1(X_0)_0$:

(1) The rational normal curve has trivial modular stratum, as computed in Example 3.2.7: Its defining equations are the 2×2-minors of

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} \in Mat(2, 4, \mathbb{C}\{x_0, \ldots, x_4\})$$

are homogeneous of degree two, but there are no elements of this degree in its T^1: A $\mathbb{C}$-basis is given by

$$\left\{ \begin{pmatrix} 0 \\ -x_3 \\ 0 \\ x_2 \\ 0 \\ x_0 \end{pmatrix}, \begin{pmatrix} x_3 \\ 0 \\ 0 \\ x_1 \\ x_0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_3 \\ 0 \\ -x_2 \\ x_1 \end{pmatrix}, \begin{pmatrix} x_4 \\ x_3 \\ x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

for instance. Thus in this example the condition imposed by the Euler relation already implies that the modular stratum has to be trivial.

(2) On the other hand the determinantal singularity of Example 3.2.8 was defined by homogeneous polynomials of degree two as well, but here we obtain a basis of T^1 whose 25 elements all have the same degree two. Thus, in this case the Euler relation is satisfied for every element.

There are a lot more non-trivial derivations that one has to take into account, but here it is possible to find lifts of all of them up to second order, giving the result of the SINGULAR-computation which implies that in this example the tangent space of the modular stratum equals the whole tangent space of the base space of the semi-universal deformation.

Let us now show what can happen for quasihomogeneous complete intersections in dimension zero. In [Nik03], a great number of modular strata for such subgerms of $(\mathbb{C}^2, 0)$ had been computed. The structure of the module of derivations of such singularities can become much more complicated, see [AM92], for instance. We present some examples illustrating how the assertion of Proposition 2.4.2 becomes completely wrong in this situation:

Example 4.3.3.

(1) The modular stratum can be singular and decompose into several components. As an example, the modular stratum of $V((x^2y^2, x^4 + y^8), 0)$ is given by the family

$$F_M = (x^2y^2 + s \cdot x^3, x^4 + y^8 + t \cdot xy^6)$$

over the base $\mathbb{C}\{s, t\}/(s \cdot t)$.

A somewhat more complicated example of a similar kind is given by $I := (x^2y^5, x^8 + y^4) \subseteq \mathbb{C}\{x, y\}$: In this case, we obtain the maximal modular family

$$F_M = (x^2y^5 + s_2 \cdot x^6y^3 + s_3 \cdot x^8y^2 + s_4 \cdot x^{12}, x^8 + y^4 + s_2 \cdot \tfrac{6}{5}x^4y^2 + s_1 \cdot x^6y)$$

over the subspace of $(\mathbb{C}^4, 0)$ defined by $s_1 \cdot (s_1 - \frac{8}{5}s_3)$. So in this case the modular stratum has a two-dimensional singular locus, being the union of two hyperplanes in $\mathbb{C}^4$ meeting in a plane.

(2) In addition, it is no longer true that M_{X_0} is a reduced germ: As an example, consider the germ defined by $(x^2y^3, x^8 + y^4) \subseteq \mathbb{C}\{x, y\}$. In this case, we get e.dim$(M_{X_0}) = 2$, with defining ideal $(t) \cap (s, t^2) \subseteq \mathbb{C}\{s, t\}$.

Similarly, a maximal modular family of the germ $V((x^2y^7, x^7 + y^7), 0) \subseteq (\mathbb{C}^2, 0)$ is given by

$$F_M = (x^2y^7 + s_2 \cdot y^9 + s_3 \cdot xy^8 + s_4 \cdot x^6y^3, x^7 + y^7 + s_1 \cdot x^5y^2 + s_4 \cdot 7x^4y^3),$$

and the modular stratum decomposes into a two-dimensional component $V(s_1, s_4)$ and an embedded point in $(\mathbb{C}^4, 0)$.

(3) Nevertheless, for many examples the situation stays the same. For instance, take $X_0 := (V(I), 0)$ with $I := (x^2y^4, x^4 + y^8) \subseteq \mathbb{C}\{x, y\}$. Then a maximal modular deformation is defined by the family

$$F_M = (x^2y^4 + s_1 \cdot y^8 + s_2 \cdot xy^6 + s_3x^3y^2, x^4 + y^8 + s_4xy^6)$$

over the smooth base $(\mathbb{C}^4, 0)$ – with respect to both deformation functors under consideration.

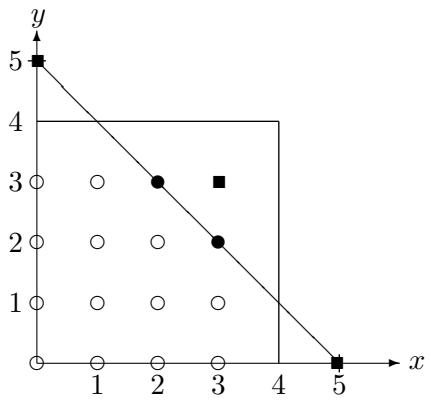
The examples presented above and in the previous sections illustrate that the modular stratum of an isolated singularity X_0 is in general non-reduced or singular, when X_0 is not a quasihomogeneous complete intersection of positive dimension. In particular, Proposition 2.4.2 becomes immediately wrong when dropping parts of the assumptions on X_0.

One might now conjecture that Proposition 2.4.2 admits a converse in the following sense: If X_0 is an isolated complete intersection of positive dimension such that the modular stratum is reduced and smooth, then X_0 is quasihomogeneous. However, this proves wrong, as the following example shows:

Example 4.3.4. Let $f := x^5 + y^5 + x^3y^3 \in \mathbb{C}\{x, y\}$ define the plane curve X_0. Then f is not quasihomogeneous as $\mu(X_0) = 16 > 15 = \tau(X_0)$, but we obtain a maximal modular family

$$F_M = f + s \cdot x^2y^3 + t \cdot x^3y^2$$

over the smooth base with local ring $\mathbb{C}\{s, t\}$. Its Newton diagram looks like this:



In general, for non-quasihomogeneous germs X_0 there is little knowledge about the structure of the module of derivations $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X_0}) \simeq T^0(X_0)$. One general fact, due to Scheja and Wiebe, is:

Proposition 4.3.5. *Let A be an analytic algebra defining a reduced isolated complete intersection singularity. If there exists a non-nilpotent derivation $\delta \in \mathrm{Der}_{\mathbb{C}}(A)$, then there exists a positive grading on A*[‡].

Proof. [SW77, section 4]. ∎

Thus we obtain the following interpretation on the modular stratum, which has already been stated in [Pal90b] for hypersurfaces and is some sort of a converse to Remark 4.3.2:

Corollary 4.3.6. *Suppose X_0 is a reduced isolated complete intersection singularity which is not quasihomogeneous. Then M_{X_0} is non-trivial.*

Proof. By Proposition 4.3.5, all derivations in $T^0(X_0)$ are nilpotent so, by Engel's theorem, there is a non-zero element $g \in T^1(X_0)$ vanishing under the action of $T^0(X_0)$ on $T^1(X_0)$. Hence $T(M_{X_0})$ and thus M_{X_0} is non-trivial. ∎

However, in many of the examples of section 4.2 the modular stratum is non-trivial but we only obtain an Artinian germ, since it is only possible to lift the vector fields in $T^0(X_0)$ infinitesimally: The exceptional uni-modular and T-series singularities (except the ones at the beginning of that series) are instances of this phenomenon.

4.4 Semi-quasihomogeneous hypersurfaces

We now specialize down our discussion to hypersurfaces. For quasihomogeneous ones, we have a complete answer concerning their modular strata, so it is a natural next step to ask what the situation is like when considering semi-quasihomogeneous hypersurfaces instead. In this class, the simplest types of singularities are those where the Tjurina number drops by one compared to the Milnor number. For such singularities, computations lead to the following

Conjecture 4.4.1. Suppose $f \in \mathcal{O}_n$ defines a semi-quasihomogeneous hypersurface singularity X_0 such that $\tau(X_0) = \mu(X_0) - 1$. In addition, let $\tilde{X}_0$ be the germ defined by the quasihomogeneous initial form of f. Then

(1) $\mathrm{e.dim}(M_{X_0}) \leq \mathrm{e.dim}(X_0) + \dim_{\mathbb{C}} T^1(\tilde{X}_0)_0$ and

(2) $\dim(M_{X_0}) = \dim(M_{\tilde{X}_0}) = \dim_{\mathbb{C}} T^1(\tilde{X}_0)_0$.

For suitably general f, equality holds in the first formula. As before, $T^1(\tilde{X}_0)_0$ denotes the eigenspace of 0 of the Euler operator.

We are not able to give a proof of this conjecture at the moment, but at least we can give a proof of its first part in case f is of a special form:

[‡]Here we say that δ is non-nilpotent if its linear part defines a non-nilpotent linear operator on the vector space $\mathfrak{m}_A/\mathfrak{m}_A^2$.

Proposition 4.4.2. *Let* $f := x_1^{p_1} + \ldots + x_n^{p_n} + x_1^{p_1-2} \cdots x_n^{p_n-2} \in \mathbb{C}\{x_1, \ldots, x_n\}$, *i. e.* f *is semi-quasihomogeneous of Hesse type and its initial form* $f_0 := x_1^{p_1} + \ldots + x_n^{p_n}$ *is a Brieskorn polynomial. Suppose* $p_i \geq 4$ *for all* i. *Then*

$$\mathrm{e.dim}(M_{X_0}) \leq \mathrm{e.dim}(X_0) + \dim_{\mathbb{C}} T^1(\tilde{X}_0)_0,$$

where X_0 *is defined by* f *and* $\tilde{X}_0$ *by* f_0. *For suitably general* f, *equality holds in this formula.*

Proof. The argument is similar to the proof of Proposition 4.2.1: Since $\mu(X_0) = \tau(X_0) - 1$ it follows that $(J(f) : f) = \mathfrak{m}_{\mathcal{O}_n}$. Thus it suffices to find derivations $\delta_1, \ldots, \delta_n \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_n)$ with the property $\delta_i(f) = x_i f$ and to determine their action on $T^1(X_0)$, in order to compute the tangent space to the modular stratum. By symmetry we only need to find δ_1, the others are obtained by permuting the variables. So consider the derivation

$$\delta_1 := \left[\frac{x_1^2}{p_1} - \frac{u}{p_1}\prod_{j=2}^{n} x_j^{p_j-2}\right]\frac{\partial}{\partial x_1} +$$

$$\sum_{i=2}^{n}\left[\frac{x_1 x_i}{p_i} + (-1)^i \frac{u}{p_i}\prod_{j=1}^{i-1}(1-\tfrac{2}{p_j})\cdot x_1^{(i-1)(p_1-2)-1}\cdot\prod_{j=2}^{i-1} x_j^{(i-1)(p_j-2)-1}\cdot x_i^{(i-1)p_i-2i+1}\cdot\prod_{j=i+1}^{n} x_j^{i(p_j-2)}\right]\frac{\partial}{\partial x_i},$$

where $u \in \mathcal{O}_n^*$ is defined as

$$\frac{n-1-2\cdot\sum_{i=1}^{n}\frac{1}{p_i}}{1+(-1)^{n-1}\prod_{j=1}^{n}\left[(1-\frac{2}{p_j})x_j^{(n-1)p_j-2n}\right]}.$$

By plugging in f one checks that $\delta_1(f) = x_1 f$. Now consider the following monomial basis of $T^1(X_0)$:

$$B := \{x_1^{\alpha_1}\cdots x_n^{\alpha_n} : \alpha_i \leq p_i - 1 \text{ for all } i\} \setminus \{x_1^{p_1-2}\cdots x_n^{p_n-2}\}.$$

Then $\delta_1(x^\alpha) - x_1 x^\alpha = (\sum_{i=1}^{n}\frac{\alpha_i}{p_i} - 1)x_1^{\alpha_1+1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} +$ higher order terms, so this expression vanishes in $T^1(X_0)$ in the following situations:

- x^α has weighted degree one with respect to the weight vector $(\frac{1}{p_1}, \ldots, \frac{1}{p_n})$, since one checks that in this case all further terms vanish modulo $(f, J(f))$ as well,
- $\alpha_1 = p_1 - 2$ or
- $\alpha_1 = p_1 - 3$ and $\alpha_i = p_i - 2$ for all $i \geq 2$.

The analogous conditions are obtained for $\delta_2, \ldots, \delta_n$, so the only elements vanishing on the action of all δ_i are

$$\text{a)}\quad x^\alpha : \sum\frac{\alpha_i}{p_i} = 1, \qquad\qquad \text{b)}\quad \frac{x_1^{p_1-2}\cdots x_n^{p_n-2}}{x_i} : 1 \leq i \leq n,$$

which gives the desired estimate on the dimension of the tangent space of M_{X_0}. For general f, the two above sets above have empty intersection, so in this case even equality holds. ∎

Remark 4.4.3.

(1) Thus we have determined as equation of the maximal modular family

$$F_M = f + \sum_{i=1}^{n} s_i \frac{x_1^{p_1-2} \cdot \ldots \cdot x_n^{p_n-2}}{x_i} + \sum_j t_j g_j,$$

with g_j the monomials in a $\mathbb{C}$-basis of $T^1(X_0)$ of weighted degree 1. Of course, the directions corresponding to s_i in the modular stratum can occur only infinitesimally in general, because these parameters do not yield τ-constant deformations – except for special cases like the singularity $x^5+y^5+x^3y^3$ of Example 4.3.4, where the monomials $\frac{x^3y^3}{x}$ and $\frac{x^3y^3}{y}$ already lie on the diagonal. So this also explains why this semi-quasihomogeneous singularity has a smooth modular stratum.

(2) For $p_i < 4$ the calculation from above becomes wrong: Take, for instance, $f := x^{10} + y^3 + x^8y \in \mathbb{C}\{x, y\}$. Then

$$F_M = f + s \cdot x^9 + t \cdot (x^8 + \tfrac{4}{5}x^6y)$$

is the equation of a maximal modular family, with base $\mathbb{C}\{s, t\}/(s, t)^2$. Nevertheless this result still agrees with the conjecture on the (embedding) dimension of the modular stratum. Note that in this example all monomials occurring in the modular family lie below the Newton boundary of the original polynomial f.

For more general semi-quasihomogeneous hypersurfaces, where the difference $\mu - \tau$ becomes bigger, the structure of the modular stratum becomes more and more complicated to describe. For instance, in the examples considered so far we only encountered zero-dimensional embedded components (if any) – cf. also our list of modular strata for unimodal singularities. By contrast, in the following example we find an embedded component of positive dimension:

Example 4.4.4. Consider the plane curve given by $f := x^6+y^6+x^3y^4 \in \mathbb{C}\{x, y\}$. A maximal modular deformation is defined by the equation

$$F_M = f + s_1 \cdot x^2y^4 + s_2 \cdot x^3y^3 + s_3 \cdot x^4y^2 + s_4 \cdot x^4y^3 + s_5 \cdot xy^4,$$

and the ideal of the modular stratum is $(s_5) \cap (s_5^2, s_2, s_4, s_1 - \frac{1}{3}s_3^2)$. In this example we also have one infinitesimal direction (namely, s_5), where the corresponding monomial (xy^4) lies below the Newton boundary of the semi-quasihomogeneous function defining $(X_0, 0)$.

As some sort of a converse to the case $\mu - \tau = 1$ let us consider the situation of a semi-quasihomogeneous hypersurface where the difference of the Tjurina and the Milnor number is as big as possible (among all polynomials with the same initial form). For such singularities we can at least prove a rough lower estimate on the dimension of its modular stratum:

Proposition 4.4.5. *Let $f \in \mathcal{O}_n$ be semi-quasihomogeneous with initial form f_0 and let X_0 denote the germ defined by f. Assume that*

$$\tau(X_0) = \min\{\tau(g) : g \in \mathcal{O}_n \text{ semi-quasihomogeneous with initial form } f_0\}.$$

Denote by $\rho(X_0)$ the number of monomials of degree bigger or equal than one in a basis of the local algebra of f (with respect to weights such that f_0 has degree one). Then

$$\dim M_{X_0} \geq \rho(X_0) - (\mu(X_0) - \tau(X_0)).$$

Proof. Let $\xi : X \to S$ be a semi-universal deformation of X_0 and $\eta : \hat{X} \to T$ a semi-universal *unfolding* of X_0. We then have $T \simeq (\mathbb{C}^\mu, 0)$. Let $T' \subseteq T$ be a subspace such that the restriction of η to T' gives a maximal τ-constant unfolding. Then, by passing to representatives of the germs under consideration, we may consider $\eta_{|T'}$ as an affine family, and its base T' has dimension $\rho(X_0)$. It can be induced from ξ by means of some morphism φ, again we consider appropriate affine families. Since ξ is semi-universal, the fibres of this morphism are of dimension $\mu - \tau$, and its image induces a modular subgerm of S since it defines a τ-constant subfamily of ξ by construction. Therefore the dimension of a maximal modular subspace is at least $\dim T' - (\mu - \tau) = \rho(X_0) - (\mu - \tau)$. ∎

So far we are not able to prove precise statements concerning the modular stratum of arbitrary semi-quasihomogeneous hypersurfaces. We thus finish by formulating a conjecture which is a variant of Conjecture 4.4.1 for certain classes of semi-homogeneous plane curves. Consider a semihomogeneous function $f \in \mathbb{C}\{x, y\}$ such that its homogeneous initial form f_0 is a form of degree d. Functions of this type have been studied in [Mar93] where coarse moduli spaces of such germs with fixed d and τ are constructed..

Conjecture 4.4.6. Let $f \in \mathbb{C}\{x, y\}$ be semi-homogeneous of degree d with homogeneous initial form f_0. Suppose that

- $f = f_0 + \lambda \cdot x^\alpha y^\beta$, where $x^\alpha y^\beta$ is a monomial of degree bigger than d in a basis of the local algebra of f, and
- we impose the further condition $2\alpha + \beta, \alpha + 2\beta > 2d - 1$.

Then

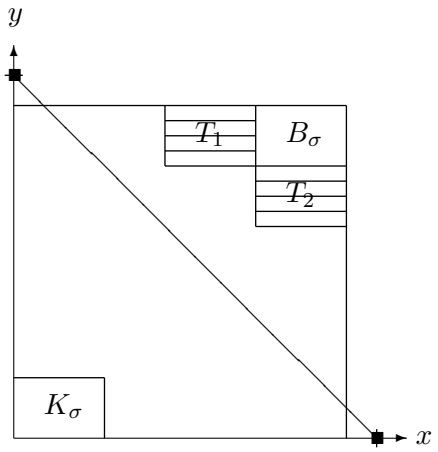
$$\text{e.dim}(M_{V(f),0}) = d - 3 + 2 \cdot (\mu(f) - \tau(f)).$$

To justify this formula, note the following: We know that the homogeneous germ $(V(f_0), 0)$ has $(d-3)$-dimensional modular stratum since this is the number of basis elements of degree d in a basis of its Tjurina algebra, so we obtain this number by Proposition 2.4.2. These elements correspond to the basis monomials on the diagonal of the Newton diagram of f, giving τ-constant deformations of $X_0 := (V(f), 0)$ as well.

In addition, consider sequence (2.3.1) of Proposition 2.3.2:

$$0 \to T^0_\bullet(X_0) \xrightarrow{\varepsilon} \mathbb{C}\{x, y\}/J(f) \xrightarrow{m_f} \mathbb{C}\{x, y\}/J(f) \xrightarrow{\pi} T^1(X_0) \to 0,$$

and recall that $\dim_\mathbb{C} T^0_\bullet(X_0) = \tau$. Thus, $\sigma := \mu - \tau$ equals the codimension of $T^0_\bullet(X_0)$ in the local algebra of f. Suppose we find a basis of $T^0_\bullet(X_0)$ of monomials $h_1, \ldots, h_\tau$ which belong to derivations $\delta_1, \ldots, \delta_\tau \in \text{Der}_\mathbb{C}(\mathbb{C}\{x, y\})$, i. e. $\delta_i(f) = h_i f$, and a set K_σ of monomials extending them to a basis of $\mathbb{C}\{x, y\}/J(f)$. We end up with a picture like this:



The monomials corresponding to the big square give a basis of the local algebra of f, $B \setminus B_\sigma$ induces a basis of $T^1(X_0)$, and $B \setminus K_\sigma = \{h_1, \ldots, h_\tau\}$. Then the monomials in the two shaded regions T_1, T_2 of the above picture vanish under the action of $T^0_\bullet$, thus they give additional $2 \cdot \#B_\sigma = 2 \cdot (\mu - \tau)$ tangent directions for the modular stratum, and if f is general, we cannot achieve $[\delta_i, b] = \delta_i(b) - h_i b = 0 \in T^1(X_0)$ for any other $b \in T^1(X_0)$ (linearly independent from those found before corresponding to T_1 and T_2).

The second condition from above now ensures that T_1 and T_2 do not intersect with the diagonal, which implies the above formula.

We close this discussion by illustrating these considerations by means of some examples:

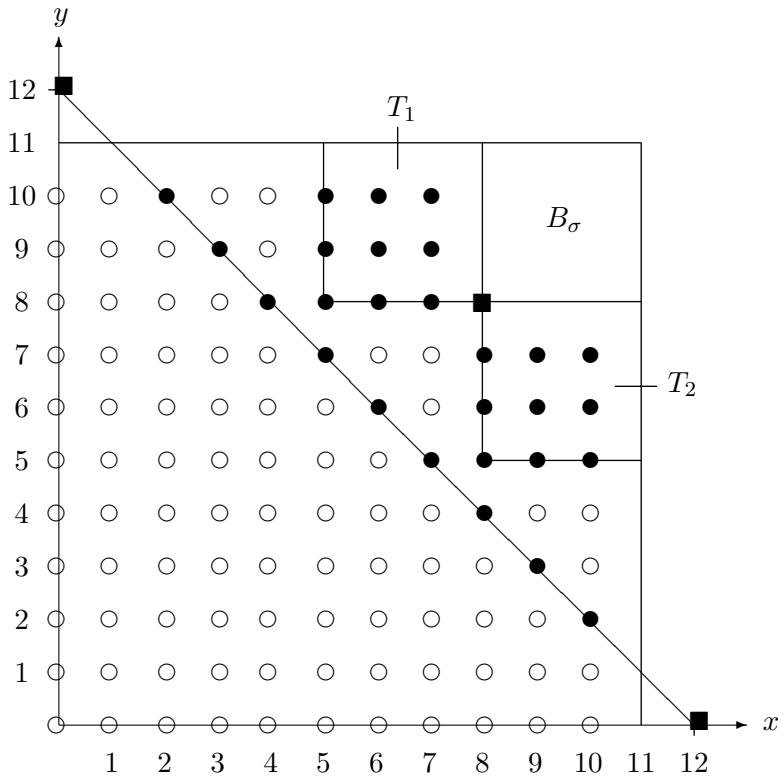
Example 4.4.7. Take $d := 12$, i. e. $f_0 := x^{12} + y^{12} \in \mathbb{C}\{x, y\}$ with Milnor number 121. The following table lists some examples of semi-homogeneous polynomials with the same initial form $x^{12} + y^{12}$:

f	$\mu - \tau$	e.dim(M_{X_0})
$f + x^{10}y^{10}$	1	$11 = 9 + 2 \cdot 1$
$f + x^{10}y^9$	2	$13 = 9 + 2 \cdot 2$
$f + x^{10}y^8$	3	$15 = 9 + 2 \cdot 3$
$f + x^{10}y^7$	4	$17 = 9 + 2 \cdot 4$
$f + x^9y^9$	4	$17 = 9 + 2 \cdot 4$
$f + x^9y^8$	6	$21 = 9 + 2 \cdot 6$
$f + x^8y^8$	9	$27 = 9 + 2 \cdot 9$

For instance, let us consider $f + x^8y^8$ with Tjurina number 112. In accordance with Conjecture 4.4.6, its modular stratum has embedding dimension $27 = (12 - 3) + 2 \cdot (121 - 112)$, and the corresponding modular family is given as

$$\begin{aligned} F_M = f &+ s_1x^{10}y^2 + s_2x^9y^3 + \ldots + s_9x^2y^{10} \\ &+ s_{10}x^8y^5 + s_{11}x^9y^5 + s_{12}x^{10}y^5 + s_{13}x^8y^6 + s_{14}x^9y^6 \\ &\qquad + s_{15}x^{10}y^6 + s_{16}x^8y^7 + s_{17}x^9y^7 + s_{18}x^{10}y^7 \\ &+ s_{19}x^5y^8 + s_{20}x^5y^9 + s_{21}x^5y^{10} + s_{22}x^6y^8 + s_{23}x^6y^9 \\ &\qquad + s_{24}x^6y^{10} + s_{25}x^7y^8 + s_{26}x^7y^9 + s_{27}x^7y^{10}. \end{aligned}$$

The picture of the Newton diagram and the respective basis monomials looks like this:



Appendix A

The SINGULAR-library modular.lib

```
LIBRARY:  modular.lib  PROCEDURES TO COMPUTE THE MODULAR STRATUM
AUTHORS:  Bernd Martin, email: martin@math.tu-cottbus.de
          Tobias Hirsch, email: hirsch@math.tu-cottbus.de

OVERVIEW:
  A library to compute modular strata of isolated singularities with respect
  to the usual deformation functor as well as to the functor of deformations
  with section.
  tangmodular(f) computes the tangent space of the modular stratum, i. e. the
  first-order approximation of a maximal modular family inside a semi-universal
  family of the singularity defined by f.
  modular(f,d) computes the modular stratum of the singularity up to order d
  by determining and killing the obstructions to lift modularity.

PROCEDURES:

 tangmodular(f[,s]);    tangent space of modular stratum of ideal f
 modular(f,d[,s]);      modular family of iso. singularity f up to degree d

////////////////////////////////////////////////////////////////////////////

tangmodular(ideal I,list #)
USAGE:    tangmodular(f,d[,s]); f ideal, d integer
ASSUME:   ordering is local,
          f is an ICIS when computing w.r.t deformations with section
RETURNS:  list of two rings:
          [1]: embedding ring of the modular stratum of f,
          [2]: embedding ring of total space of the modular family, F0+F1
               inside this ring give the equations of the modular family
               except in the case when the modular stratum is trivial -
               in which nothing is returned.
          This gives the tangent space and the infinitesimal modular family
          for f - higher order jets of the modular stratum are computed in
```

```
          the procedure modular.
          If an additional argument s is given and non-zero, then computations
          ware performed w.r.t. the functor of deformations with section
EXAMPLE:  example tangmodular; shows an example.

////////////////////////////////////////////////////////////////////////////

modular(ideal f, int d, list #)
USAGE:    modular(f,d[,s]); f ideal, d integer
ASSUME:   ordering is local,
          f is an ICIS when computing w.r.t deformations with section
RETURNS:  list of two rings:
          [1]: embedding ring of the modular stratum of f, containing the
               ideals J(k) defining the k-th jet of the modular stratum
               for 2<=k<=d;
          [2]: embedding ring of total space of the modular family, F0+F1
               inside this ring give the equations of the modular family,
               except in the case when the modular stratum is trivial - in which
               nothing is returned.
          If an additional argument s is given and non-zero, then computations
          are performed w.r.t. the functor of deformations with section
EXAMPLE:  example modular; shows an example.
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Bibliography

[AGLV98] V. I. Arnol'd, V. V. Goryunov, O. V. Lyashko, and V. A. Vasil'ev, *Singularity theory I*, Springer-Verlag, Berlin, 1998.

[AGZV85] V. I. Arnol'd, S. M. Guseĭn-Zade, and A. N. Varchenko, *Singularities of differentiable maps Vol. I*, Monographs in Mathematics, vol. 82, Birkhäuser Boston Inc., Boston, MA, 1985.

[Ale82] A. G. Aleksandrov, *Normal forms of one-dimensional quasihomogeneous complete intersections*, Mat. Sb. (N.S.) **117** (1982), no. 1, 3–31, *english translation:* Math. USSR-Sb. **45** (1983), no. 1, 1–30.

[Ale85] ———, *Cohomology of a quasihomogeneous complete intersection*, Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), no. 3, 467–510, *english translation:* Math. USSR-Izv. **26** (1986), no. 2, 437–477.

[Ale00] ———, *Modular space for complete intersection curve-singularities*, Finite or infinite dimensional complex analysis (Fukuoka, 1999), Lecture Notes in Pure and Appl. Math., vol. 214, Dekker, New York, 2000, pp. 1–19.

[AM92] A. G. Aleksandrov and B. Martin, *Derivations and deformations of Artinian algebras*, Beiträge Algebra Geom. **33** (1992), 115–130.

[And74] M. André, *Homologie des algèbres commutatives*, Die Grundlehren der mathematischen Wissenschaften, vol. 206, Springer-Verlag, Berlin, 1974.

[Bin80] J. Bingener, *Offenheit der Versalität in der analytischen Geometrie*, Math. Z. **173** (1980), no. 3, 241–281.

[Bri79] E. Brieskorn, *Die Hierarchie der* 1*-modularen Singularitäten*, Manuscripta Math. **27** (1979), no. 2, 183–219.

[Buc81] R. O. Buchweitz, *Contributions à la théorie des singularités*, Thèses, Université Paris VII, 1981.

[dJP00] T. de Jong and G. Pfister, *Local analytic geometry*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 2000.

[Eis95] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, Berlin, 1995.

[FK00] A. Frühbis-Krüger, *Moduli Spaces for Space Curve Singularities*, Dissertation, Universität Kaiserslautern, 2000.

[Fle78] H. Flenner, *Über Deformationen holomorpher Abbildungen*, Habilitationsschrift, Universität Osnabrück, 1978.

[Fle81] H. Flenner, *Ein Kriterium für die Offenheit der Versalität*, Math. Z. **178** (1981), no. 4, 449–473.

[GHP97] G.-M. Greuel, C. Hertling, and G. Pfister, *Moduli spaces of semiquasihomogeneous singularities with fixed principal part*, J. Algebraic Geom. **6** (1997), no. 1, 169–199.

[GM02] G.-M. Greuel and B. Martin, `sing.lib`, a SINGULAR 2.0 library for computing invariants of singularities, 2002.

[GP94] G.-M. Greuel and G. Pfister, *Moduli for singularities*, Singularities (Lille, 1991), London Math. Soc. Lecture Note Ser., vol. 201, Cambridge Univ. Press, Cambridge, 1994, pp. 119–146.

[GP02] G.-M. Greuel and G. Pfister, *A* SINGULAR *Introduction to Commutative Algebra*, Springer, Berlin, 2002.

[GPS02] G.-M. Greuel, G. Pfister, and H. Schönemann, SINGULAR 2.0, A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2002, `http://www.singular.uni-kl.de`.

[GR71] H. Grauert and R. Remmert, *Analytische Stellenalgebren*, Die Grundlehren der mathematischen Wissenschaften, vol. 176, Springer-Verlag, Berlin, 1971.

[Gra72] H. Grauert, *Über die Deformation isolierter Singularitäten analytischer Mengen*, Invent. Math. **15** (1972), 171–198.

[Hau83] H. Hauser, *An algorithm of construction of the semiuniversal deformation of an isolated singularity*, Singularities, Part 1 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, R.I., 1983, pp. 567–573.

[HM03] T. Hirsch and B. Martin, *Deformations with section: Cotangent cohomology, flatness conditions and modular subgerms*, preprint M-03/2003, Institut für Mathematik, BTU Cottbus, 2003, `math.CV/0306281`.

[HM04] T. Hirsch and B. Martin, `modular.lib`, a SINGULAR 2.0 library for computing the modular stratum, 2004.

[Ill71] L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, vol. 239, Springer-Verlag, Berlin, 1971.

[Kos88] S. Kosarew, *Ein allgemeines Kriterium für das formale Prinzip*, J. Reine Angew. Math. **388** (1988), 18–39.

[Kos91] ———, *Maximal modular subspace in semi-universal deformation of holomorphic mapping germs*, Analyse complexe (Bucharest, 1989), Rev. Roumaine Math. Pures Appl. **36** (1991), no. 5-6, 231–252.

[KS77] B. Kramm and H.-W. Schuster, *Über die Cohomologie analytischer Algebren*, Rev. Roumaine Math. Pures Appl. **22** (1977), no. 5, 655–668.

[KS90] S. Kosarew and H. Stieber, *A construction of maximal modular subspaces in local deformation theory*, Abh. Math. Sem. Univ. Hamburg **60** (1990), 17–36.

[Lod98] J.-L. Loday, *Cyclic homology*, Grundlehren der Mathematischen Wissenschaften, vol. 301, Springer-Verlag, Berlin, 1998.

[Loo84] E. J. N. Looijenga, *Isolated singular points on complete intersections*, London Mathematical Society Lecture Note Series, vol. 77, Cambridge University Press, Cambridge, 1984.

[LP88] O. A. Laudal and G. Pfister, *Local moduli and singularities*, Lecture Notes in Mathematics, vol. 1310, Springer-Verlag, Berlin, 1988.

[LS67] S. Lichtenbaum and M. Schlessinger, *The cotangent complex of a morphism*, Trans. Amer. Math. Soc. **128** (1967), 41–70.

[Man01] M. Manetti, *Deformations of singularities via differential graded lie algebras*, electronic notes, University of Rome La Sapienza, 2001, available via `http://venere.mat.uniroma1.it/people/manetti/`.

[Mar93] B. Martin, *Coarse moduli space of ordinary multiple points in the plane*, Math. Z. **212** (1993), no. 4, 479–504.

[Mar99] ———, *Computing versal deformations with* SINGULAR, Algorithmic algebra and number theory (Heidelberg, 1997), Springer, Berlin, 1999, pp. 283–293.

[Mar02a] ———, *Algorithmic computation of flattenings and of modular deformations*, J. Symbolic Comput. **34** (2002), no. 3, 199–212.

[Mar02b] B. Martin, `deform.lib`, a SINGULAR 2.0 library for computing miniversal deformations of singularities and modules, 2002.

[Mar03] ———, *Modular Deformations and Space Curve Singularities,* Algebraic Geometry and Singularities (Sevilla 2001), Rev. Mat. Iberoamericana **19** (2003), no. 2, 613–621.

[Mat86] H. Matsumura, *Commutative ring theory*, Cambridge studies in advanced mathematics, vol. 8, Cambridge University Press, Cambridge, 1986.

[MvS01] D. Mond and D. van Straten, *The structure of the discriminant of some space-curve singularities*, Q. J. Math. **52** (2001), no. 3, 355–365.

[New78] P. E. Newstead, *Introduction to moduli problems and orbit spaces*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 51, Tata Institute of Fundamental Research, Bombay, 1978.

[Nik03] D. Nikitchenko, *Berechnung des Modularstratums für Singularitäten von vollständigen Durchschnitten und Raumkurven*, Diplomarbeit, Brandenburgische Technische Universität Cottbus, 2003.

[Pal76] V. P. Palamodov, *Deformations of complex spaces*, Uspehi Mat. Nauk **31** (1976), no. 3, 129–194, *english translation:* Russian Math. Surveys **31** (1976), no. 3, 129–197.

[Pal78] ———, *Moduli in versal deformations of complex spaces*, Variétés analytiques compactes (Colloq., Nice, 1977), Lecture Notes in Mathematics, vol. 683, Springer, Berlin, 1978, pp. 74–115.

[Pal82] ———, *Cohomology of analytic algebras*, Trudy Moskov. Mat. Obshch. **44** (1982), 3–61, *english translation:* Trans. Moscow Math. Soc. **44** (1983), no. 2, 1–61.

[Pal85] ———, *Asymptotic expansions of integrals in complex and real domains*, Mat. Sb. (N.S.) **127** (1985), no. 2, 209–238, *english translation:* Math. USSR-Sb. **55** (1986), no. 1, 207–236.

[Pal90a] ———, *Deformations of complex spaces*, Several complex variables IV, Encyclopaedia of Mathematical Sciences, vol. 10, Springer-Verlag, Berlin, 1990, pp. 106–194.

[Pal90b] ———, *Tangent fields on deformations of complex spaces*, Mat. Sb. **181** (1990), no. 10, 1320–1340, *english translation:* Math. USSR-Sb. **71** (1992), no. 1, 163–182.

[Pal94] ———, *Modular deformations of complex analytic spaces*, Pure Mathematics preprint series 1, 1994, University of Oslo, Department of Mathematics, 1994.

[Pal02] ———, *Modular deformations in complex analytic gemetry*, preprint, Tel Aviv University, 2002, available via `http://www.math.tau.ac.il/~palamodo/`.

[Pop77] H Popp, *Moduli theory and classification theory of algebraic varieties*, Lecture Notes in Mathematics, vol. 620, Springer-Verlag, Berlin, 1977.

[Qui70] D. Quillen, *On the (co-) homology of commutative rings*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 65–87.

[Sai80] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291.

[Sch68] M. Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. **130** (1968), 208–222.

[Sch73] ———, *On rigid singularities*, Complex analysis, 1972 (Proc. Conf., Rice Univ., Houston, Tex., 1972), Vol. I: Geometry of singularities, Rice Univ. Studies **59** (1973), no. 1, 147–162.

[Sch77] M. Schaps, *Deformations of Cohen-Macaulay schemes of codimension* 2 *and non-singular deformations of space curves*, Amer. J. Math. **99** (1977), no. 4, 669–685.

[Ste96] J. Stevens, *Deformations of singularities*, Habilitationsschrift, Universität Hamburg, 1996.

[Ste03] J. Stevens, *Deformations of singularities*, Lecture Notes in Mathematics, vol. 1811, Springer-Verlag, Berlin, 2003.

[Sti88] H. Stieber, *Existenz semiuniverseller Deformationen in der komplexen Analysis*, Aspects of Mathematics, vol. D5, Friedr. Vieweg & Sohn, Braunschweig, 1988.

[Sti90] ———, *Über die Existenz von maximalen modularen Unterkeimen und lokalen Modulräumen in der komplexen Analysis*, Math. Z. **203** (1990), no. 2, 263–277.

[SW77] G. Scheja and H. Wiebe, *Über Derivationen in isolierten Singularitäten auf vollständigen Durchschnitten*, Math. Ann. **225** (1977), no. 2, 161–171.

[Tei77] B. Teissier, *The hunting of invariants in the geometry of discriminants*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 565–678.

[vS95] D. van Straten, *A note on the discriminant of a space curve*, Manuscripta Math. **87** (1995), no. 2, 167–177.

[Wah83] J. M. Wahl, *Derivations, automorphisms and deformations of quasihomogeneous singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 613–624.

[Wei94] C. Weibel, *An introduction to homological algebra*, Cambridge studies in advanced mathematics, vol. 38, Cambridge University press, Cambridge, 1994.