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Quantum-Based Construction of a Probability Measure

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QUANTUM-BASED CONSTRUCTION OF A PROBABILITY MEASURE

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Abstract: From Gleason's theorem [1] we know that in principle every probability measure can be expressed by Hermitian operators in a separable Hilbert space and the Born rule as part of a quantum mechanical system. However, that theorem is not constructive. For a given discrete and additive probability measure based on a σ -algebra we construct a quantum system with projectors expressing that probability measure.

Let us start with the definition of a discrete probability measure following the well-known Kolmogorow axioms.

Let be given a finite set $\Omega = \{\omega_1, \dots, \omega_m\}$ of outcomes of a random process. A σ -algebra $\Sigma \subseteq 2^\Omega$ provides a set of events closed under union and negation (realized as set complement). A mapping $P : \Sigma \rightarrow [0, 1] \subseteq \mathcal{R}$ is called an *additive, discrete probability measure* if

1. $P(\Omega) = 1$ and
2. for every countable sequence of mutual disjoint sets $\sigma_1, \dots, \sigma_n$ from Σ

$$P\left(\bigcup_{i=1}^n \sigma_i\right) = \sum_{i=1}^n P(\sigma_i)$$

hold.

Assume, a discrete and additive probability measure P on Σ is given. Next we will show how to construct a quantum system with projectors realizing the probability measure P . Here we use the Dirac notation.

A quantum system state is defined by a density matrix D . We construct our density matrix by performing following steps:

1. Take a Hilbert space \mathcal{H}^m over a orthonormal basis $B = \{|1\rangle, \dots, |m\rangle\}$ where $|i\rangle$ refers bijectively to $\omega_i \in \Omega$.
2. We construct a ket $|s\rangle \in \mathcal{H}^m$ as

$$|s\rangle = \sum_{\omega_i \in \Omega} \sqrt{P(\{\omega_i\})} |i\rangle.$$

It is obvious from the Kolmogorow axioms that the ket $|s\rangle$ has length one ($\langle s|s\rangle = 1$).

3. Now, we obtain the density matrix D by $D = |s\rangle\langle s|$. It can be easily shown that D is a Hermitian operator with trace one ($tr(D) = 1$).

Next we define a projector Pr_σ (a Hermitian operator with $Pr_\sigma \cdot Pr_\sigma = Pr_\sigma$) for every $\sigma \in \Sigma$ by

$$Pr_\sigma = \sum_{\omega_i \in \sigma} |i\rangle\langle i|.$$

Using Borns rule we obtain

$$P(\sigma) = tr(Pr_\sigma D).$$

That equivalence can be derived from

$$\begin{aligned} tr(Pr_\sigma D) &= tr\left(\left(\sum_{\omega_i \in \sigma} |i\rangle\langle i|\right) \cdot |s\rangle\langle s|\right) \\ &= \sum_{\omega_i \in \sigma} tr(|i\rangle\langle i| \cdot |s\rangle\langle s|) \\ &= \sum_{\omega_i \in \sigma} tr(\langle i|s\rangle\langle s|i\rangle) \\ &= \sum_{\omega_i \in \sigma} \langle i|s\rangle\langle s|i\rangle \\ &= \sum_{\omega_i \in \sigma} \sqrt{P(\{\omega_i\})}\sqrt{P(\{\omega_i\})} \\ &= \sum_{\omega_i \in \sigma} P(\{\omega_i\}) \\ &= P\left(\bigcup_{\omega_i \in \sigma} \{\omega_i\}\right) \\ &= P(\sigma). \end{aligned}$$

In that way, we mapped concepts of probability theory to concepts of geometry. Further relations exist:

- The probability of the conjunction of two events $\sigma_1, \sigma_2 \in \Sigma$ corresponds to the intersection of the vector subspace of their corresponding projectors. That intersection reduces to the intersection of the underlying sets of basis kets:

$$P(\sigma_1 \wedge \sigma_2) = tr(P_{\sigma_1 \wedge \sigma_2} D)$$

with

$$P_{\sigma_1 \wedge \sigma_2} = \sum_{\omega_i \in \sigma_1 \cap \sigma_2} |i\rangle\langle i|.$$

Analogously, the disjunction refers to the union and the negation to the set complement. Therefore, they obey the rules of Boolean algebra.

- For two probability measures P_1 on Σ_1 and P_2 on Σ_2 a product probability measure $P_{\sigma_1 \wedge \sigma_2}$ refers to the tensor product of their quantum mappings. Every pair of events ($\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2$) of a product probability measure over (P_1, Σ_1) and (P_2, Σ_2) are independent. The property $P_{\sigma_1 \wedge \sigma_2}(\sigma_1 \wedge \sigma_2) = P_1(\sigma_1) \cdot P_2(\sigma_2)$ is the consequence from the tensor product and the inner product: $\langle i_1 i_2 | j_1 j_2 \rangle = \langle i_1 | j_1 \rangle \langle i_2 | j_2 \rangle$.

References

- [1] GLEASON, A.: *Measures on the Closed Subspaces of a Hilbert Space*. Indiana Univ. Math. J., 6:885–893, 1957.