

Tax Effects on Asset Prices

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Abbreviations

| | |
|------|--|
| AO | Abgabenordnung |
| APV | Adjusted Present Value |
| BFH | Bundesfinanzhof |
| CAPM | Capital Asset Pricing Model |
| CARA | Constant Absolute Risk Aversion |
| COD | Cancellation of Debt |
| DCF | Discounted Cash Flow |
| EBIT | Earnings Before Interest and Taxes |
| ECF | Equity Cash Flow |
| FCF | Free Cash Flow |
| Int | Interest Payments |
| IRC | Internal Revenue Code |
| KStG | Körperschaftsteuergesetz |
| NA | Not Applicable |
| PP | Principal Payments |
| RP | Redemption Payments |
| SDF | Stochastic Discount Factor |
| TADR | Tax Adjusted Discount Rate |
| TS | Tax Savings |
| VHB | Verband der Hochschullehrer für Betriebswirtschaft |
| VTs | Value of Tax Savings |
| WACC | Weighted Average Cost of Capital |

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Abstract

In this dissertation I treat questions of asset pricing under the presence of taxes. I present four published articles. The first three articles are concerned with topics of company valuation when debt is risky. The first article analyzes the applicable discount rate for the valuation of tax savings in a simple setting without taxes on cancelled debt. The article concludes that the discount rate of tax savings is the same as the one for interest payments. Other than frequently assumed, this discount rate is not necessarily the same as the discount rate for debt as a whole. With the prioritization of interest or principal payments in case of losses on debt payments, the discount rates on interest payments and of tax savings are regularly different from the one for the overall debt issue. Only a pro rata distribution of losses on principal and interest payments generally leads to equal discount rates for interest payments, principal payments, and, therefore, also for debt payments as a whole. The second article continues to look at the valuation of tax savings. It differentiates the case with and the one without the taxation of cancelled debt. For both cases, the article derives equations for the value of tax savings as well as for risk-adjusted discount rates and WACC-like equations. A major finding is that the previous corporate finance literature on this topic usually makes the implicit assumption that cancelled debt is taxed. In this case valuation equations have a simple form since they are independent from the distribution of losses between interest and principal payments. Without the taxation of cancelled debt, the distribution of losses on interest and principal payments becomes important for the valuation procedures and a differentiation between cases such as interest prioritization, principal prioritization and pro rata loss distribution is necessary. The third article uses the findings of the first two articles and constructs equations for a de- and re-levering procedure using the expected return equations from the mean-variance CAPM. It extends the regularly used procedure that uses the assumption of risk-free debt to simple settings with risky debt. The fourth article looks at two economies, which differ only with respect to taxation - one features taxes on capital gains and one does not. The analysis leads to several extensions on prior findings on the question of the conditions under which asset prices are the same in both economies. The article provides sufficient conditions for unchanged prices for the case of a zero risk-free rate, which entails that all agents consume exactly the same in each state. Without a zero risk-free rate, prices are the same in both economies with exponential utility and normal returns and with linear marginal utility.

Chapter 1

Introduction

1.1 Tax effects on asset valuation

This dissertation contains four articles that describe tax effects on asset prices. Tax codes are usually complex and extensive, and many tax rules have significant effects on prices. Ignoring tax effects in asset pricing is likely to decrease the usefulness of a corporate valuation down to the point at which it is detrimental for any use.

For a basic example consider a tax rate τ , which, applied to a tax base X , leads to taxes at an amount of τX . For an investor that receives the after-tax cash flow $X(1 - \tau)$ the value of this cash flow should be the discounted expected after-tax cash flow. Using the price of the pre-tax cash flow for this after-tax cash flow leads to an over overvaluation of the latter one. The investor would pay for the tax as well even though this amount will not be received and cannot be consumed.¹ One can add to this example another asset that pays the same cash flow X , but is tax exempt. Now the investor not considering taxes would pay the same price for the taxed and the untaxed asset. The misvaluation becomes even more evident, and a basic law in finance, the law of one price (compare for example (Cochrane, 2005, pp.62 ff)), is violated.

The tax bases considered herein are corporate profits and capital gains. The taxes on corporate profits have the direct effect of reducing after-tax profits paid to the owners of the company. Similarly, a capital gains tax will reduce capital gains to the owner of an asset that has appreciated. Those taxes will have direct effects on reducing the tax base. But there are also indirect effects of taxes, which are caused by general equilibrium effects. A simple story here would be the example of a relative increase of the tax rate on capital gains versus the tax rate on dividends. A possible equilibrium effect may be that investors buy less assets that pay low or no dividends but are expected to appreciate more and buy more assets that pay higher dividends but are expected to

¹A note of caution here. To be precise, taxes may come back in the form of a public good or distributions and as such may have value and may not be just "burned money". However, this value is very hard to evaluate on the individual level. Thus, for corporate valuation taxes are usually treated as "burned money" (for example in the famous tax CAPM in Brennan (1970)). In turn, macroeconomic models usually include taxes and tax transfers (as for example in Sialm (2009)).

1.2. Contribution to the respective literature

appreciate less.

Through the potentially significant effects on asset pricing, taxes are an important factor in asset valuation. Due to the complexity and variety of tax rules, a model that includes all of them is unheard of. However, important features that approximate reality can be incorporated and improve the model substantially. My contribution to the asset valuation literature is exactly that. The first three articles are rooted in the works of Modigliani and Miller (1958), Modigliani and Miller (1963) and the literature that followed them. I analyze the value of corporate tax savings adding the feature of risky debt. Additionally, I analyze the effect of adding or not adding the taxation of cancelled debt. I conclude that, with a taxation of cancelled debt, the distribution of losses on interest and principal payments has no influence on asset pricing. When cancelled debt is not taxed this distribution is important and the tax savings become interest payments scaled by the tax rate. A special case is a pro-rata loss distribution on interest and principal payments. Simple valuation equations without resorting to option pricing can be derived. In the last article, I extend the work of Kruschwitz and Löffler (2009). It shows cases when two economies exist, which only differ in that one features capital gains taxes and the other does not, and which have the same asset prices. That means any payoff in the no-tax economy has the same price as the taxed payoff in the tax-economy. I turn from the mean-variance CAPM setting as in Kruschwitz and Löffler (2009) to a setting with expected utility over consumption. I establish conditions and portfolio rules for price equality in this setting, and I find new cases in which economies with equal prices exist. I will describe the contribution of each single article more in detail in Section 1.3. I will first put my work into the context of the respective finance and economics literature.

1.2 Contribution to the respective literature

Modigliani and Miller (1958) make an important breakthrough in the theory of corporate valuation. Their theorem that the capital structure, i.e., the share of debt and equity, does not influence the value of the firm, remains an important benchmark. It relies on several assumptions that are not met in practice. However, knowing those assumptions, and knowing which ones are violated gives a starting point into the inquiry of whether an optimal capital structure exists. Modigliani and Miller (1963) realize that corporate taxation and the deductibility of interest payments from pre-tax profits may add value to the firm through tax savings. More debt means more interest payments and more tax savings. Thus, tax savings provide a powerful incentive to lever the firm. The valuation of tax savings, be it theoretical or empirical remains an important research object until today. The first three articles of my dissertation analyze those tax savings under new perspectives.

Sharpe (1964) published another major breakthrough in corporate finance with what is today known as the capital asset pricing model (CAPM). Again, it is a theoretical model and its assumptions are not met in practice. But as with the theorems of Modigliani and Miller, it is an important benchmark model to explain the financial world. Apart from portfolio implications, it delivers a simple expected return equation, which shows the relationship of individual asset returns

1.2. Contribution to the respective literature

to a single risk factor - which is the return on the market portfolio. The expected return equation can easily be converted into a regression equation, which can be used for statistical analysis. The sensitivity to the market return is called beta or beta factor. In Krause (2018a), I use the CAPM expected return equation to derive equations that can be used for a de- and re-levering procedure.

The CAPM came along initially without any considerations for taxes. Brennan (1970) extended the CAPM including the various investor tax rates and taxes on corporate payouts, such as the tax rate on dividends, interest payments, and capital gains. Similar models are used until the present day to analyze asset prices with the presence of taxes. Kruschwitz and Löffler (2009) concentrate on capital gains taxes in a CAPM and explore cases for which prices may not be changed by the presence of the tax. They find a zero risk-free rate and constant absolute risk aversion to be cases in which prices may remain unchanged.² In Krause (2018b), I pick up this analysis and extend it to a different pricing framework and to some other cases.

Miller (1977) writes that the benefits from tax savings on interest payments may be greatly diminished and may even disappear when personal taxes are considered as well. Additionally, he uses equilibrium arguments of bond supply and demand to establish that firms, in the aggregate, limit the issuance of debt instruments somehow. He also admits that this does not explain leverage at the individual firm level.

Myers (1977) points to the inability of the theorems by Modigliani and Miller to explain observed capital structures. He introduces his paper by arguing that finance theory should explain why the tax savings do not lead to the firm to increasing debt as much as possible, and also what "as much as possible" actually means. Myers uses option theory to explain why firms issuing risky debt behave differently than firms issuing risk-free debt.

Brennan and Schwartz (1978) consider asset pricing including the value of tax savings. They use a continuous time framework and look at optimal leverage for a company that issues a coupon paying bond. They also include bankruptcy cost in their analysis. However, even without bankruptcy cost, they find that there is a maximum value for the tax savings below 100% leverage.

Leland (1994) continues the way of Brennan and Schwartz (1978). He uses a continuous time model to derive closed form pricing equations for firm valuation under the presence of tax savings and bankruptcy cost. Leland and Toft (1996), Leland (1998) and Goldstein et al. (2001) continue this work in more sophisticated models. The models result in optimal capital structures for companies that are below the classical 100% result. They also explain several observations in the investment grade and junk bond market.

Miles and Ezzell (1980) and Miles and Ezzell (1985) derive equations for weighted average cost of capital (WACC) including the value of tax savings. They focus on a policy in which firms maintain a constant leverage ratio. Even though it is not explicitly stated, their analysis only treats the case of risk-free debt. Harris and Pringle (1985) go on to develop risk-adjusted discount

²I use the word "may" here because the equilibrium in the tax or no-tax economy need not be unique so that a definite answer cannot be provided.

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rates including the value of tax savings.

Compared to the mentioned work of Miles and Ezzel, Sick (1990) goes further in that he includes the possibility of default in the analysis of tax savings. He derives tax-adjusted discount rates for defaultable debt under the assumption that a cancellation of debt (COD) is taxed. He finds that the structure of the derived equations is similar to the ones under risk-free debt. The assumption of a taxation of cancelled debt is picked up several times in the following literature. It is also the major research object of the second article herein. Krause and Lahmann (2017) look at several issues related to this assumption. They derive pricing equations, tax-adjusted discount rates and WACC-like discount rates with and without a taxation of a COD. With a taxation of a COD this distribution does not matter for firm valuation. They also point to important issues when there is no taxation of a COD. In this case the distribution of losses on interest and principal payments influences the valuation equation. This case is analyzed in detail in the first article presented herein. As stated in Krause and Lahmann (2016), without taxes on a COD tax savings are interest payments scaled by the tax rate. Thus, the discount rate on interest payments is also the discount rate in tax savings.

An attempt to measure the value of tax benefits is made by Graham (2000). He finds the tax benefit to be 9.7% of a firm's market value for a typical firm of his sample over 15 years looking at over 5,000 to over 6,000 firms per year. In contrast to that, the classic measure would have revealed the higher share of 13.2%. After accounting for personal taxes the tax benefit estimated by Graham (2000) shrinks to 4.3%.

More on the empirical side, Kemsley and Nissim (2002) use regression methods to estimate the value of the tax shield. They find that the value of tax shields is about 40% of debt balances and 10% of firm values. This is in line with what the study of Graham (2000) reveals.

The study of Blouin et al. (2010) asks whether tax benefits have been overestimated and whether firms do not lever enough to maximize tax savings. With a new nonparametric procedure, they find that tax benefits are much less than previously thought. Adding bankruptcy cost and debt savings not related to debt, actual capital structures seem plausible.

Furthermore, Lin and Flannery (2013) analyze the 2003 tax cut that only affected individual investors. They find that the cut decreased leverage of the firms by about five percent. They conclude that personal taxes strongly affect firms' optimal leverage.

Arzac and Glosten (2005) elaborate on discount rates for company valuation when leverage is constant. The issue here is that no assumption on the treatment of a COD is made. They also make statements about discount rates when debt is risky. They want to use a rate that includes a risk premium in the equation to value tax savings. However, as Krause and Lahmann (2017) show, one has to reverse engineer the underlying assumptions to make their statement hold, and there are several possibilities for them to hold.

Kruschwitz et al. (2005) include defaultable debt into the discounted cash flow model. They find that the valuation equations with and without default risk are identical. Some of their im-

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portant assumptions are that cancelled debt is taxed, interest payments are tax deductible, the tax authorities always receive the taxes the firm owes, and that the financing policy is exogenously given. The last item means that after a default, the firm continues with the same financing policy as before.

Cooper and Nyborg (2008) analyze tax-adjusted discount rates looking especially at the assumption of a taxation of a COD. They state to have identified the assumption that is in line with discounting interest tax savings at the cost of debt. The assumption is that there is no tax on a COD. They hold that assumption against the one used in Sick (1990), who obtains that tax savings should be discounted at the risk-free rate when a COD is taxed. In Krause and Lahmann (2017), my coauthor and I show that there are several issues with those analyses and the statements made. The cost of debt needs a precise definition. Naturally, one would expect it to be the conditional expected return on debt. However, in the analysis it becomes more likely that it is the coupon rate for single-period debt or the yield rate. Cooper and Nyborg (2008) point at that but there are still restrictions on that observation. Krause and Lahmann (2017) show that the result depends on the pro rata loss distribution between interest and principal payments. Cooper and Nyborg (2008) use a binomial model, in which one state always leads to a complete loss. This is a special case of a pro rata loss distribution. The model choice of Cooper and Nyborg (2008) does not allow to see the dependence of their result on the loss distribution. Furthermore, the risk-free rate as a discount rate in Sick (1990) is not the discount rate of expected tax savings but just a parameter in the valuation equation.

Molnár and Nyborg (2013) again look at tax-adjusted discount rates, but now they consider partial losses instead of complete losses in a default. The model remains a two-state model. They derive an equation for a tax-adjusted discount rate including personal tax rates. Their equation provides some generality because it includes personal taxes and a parameter for recoveries in the default state. As in Molnár and Nyborg (2013), Krause and Lahmann (2016) also look at the cases of an interest and a principal prioritization. However, this analysis is not restricted to a binomial model, but uses a multistate model to overcome the weaknesses as mentioned above. It also uses findings from option pricing as in Coval and Shumway (2001) to characterize discount rates on tax savings instead of applying a single parameter to express losses.

Koziol (2014) derives a WACC that accounts for default risk and bankruptcy cost. He finds that significant upward shifts from the usual WACC are possible when the additional default components are included.

Kruschwitz et al. (2011b) find inconsistencies in the de- and re-levering procedure of using the Modigliani and Miller adjustment equations. In turn they agree to the Miles and Ezzel equations. A quick discussion arose with Meitner and Streitferdt (2011), who claim that the Modigliani and Miller equation is valid under the assumptions given in Kruschwitz et al. (2011b). In their reply, Kruschwitz et al. (2011a) find that their prior result still holds true. However, they find the cause of the discussion to be in the different definition of the cost of capital used in Meitner and Streitferdt

1.3. Description and integration of the different articles

(2011). They reject the definition used in Meitner and Streitferdt (2011) as not being useful and economically not interpretable. Kruschwitz et al. (2011b) provide and discuss the Miles and Ezzel equation, their Equation (22), which I also restate in Krause (2018a). However, I do not only cover the case of taxes on cancelled debt but continue the analysis for the case of untaxed cancelled debt and three different cases of loss distributions between interest and principal payments.

There are several more studies dedicated to valuing tax savings. For example, Liu (2009) analyzes several misconceptions of the finance literature related to the tax shield. Especially, Liu (2009) divides into or "slices" the value of the tax shield into an "earned" and an "unearned" part, in which the first one is the actual value of the tax shield and the second one the value of leverage return. Qi (2011) responds to the paper of Liu (2009) and refutes the results in Liu (2009). Above all, Liu (2009) refutes the Modigliani and Miller results, and Qi (2011) restores those results. Furthermore, Couch et al. (2012) take on a barrier options approach to tax shield valuation. Risky debt is included in the analysis. They derive practical valuation equations for the scenarios of constant debt, delayed debt, as well as debt refinancing.

Having classified my work into the respective literature, I will continue to describe the articles that comprise this dissertation more in detail. I will also explain, how they relate to one another.

1.3 Description and integration of the different articles

Table 1.1 summarizes the four articles that will follow after this introductory chapter. All of them are about asset pricing, and all of them study how assets are priced under the influence of a certain kind of tax. The tax rate used is always deterministic.

Table 1.1: Article overview

| Title | Year | Journal | Authors |
|--|------|---|-----------------------------------|
| Reconsidering the appropriate discount rate for tax shield valuation | 2016 | Journal of Business Economics | Marko Krause Alexander Lahmann |
| Valuation effects of taxes on debt cancellation | 2017 | Quarterly Review of Economics and Finance | Marko Krause Alexander Lahmann |
| De- and re-levering betas with risky debt revisited | 2018 | Business Research | Marko Krause |
| Effects of a capital gains tax on asset pricing | 2018 | Business Research | Marko Krause |

The first three articles presented in Table 1.1 are more corporate finance related. They have several common assumptions. The analyses are about firm valuation. The firms have taken out some sort of risky debt. No general equilibrium effects are considered. As in Modigliani and Miller (1963) and a lot of the following literature only corporate taxes are considered. Interest payments are deducted from taxable income so that tax savings are generated through taking out interest-paying debt. Additionally, in the event of a default, the tax savings are not necessarily fully lost.

1.3. Description and integration of the different articles

Partial interest payments lead to partial tax savings, an assumption used for example in Molnár and Nyborg (2013). Furthermore, those articles study the effects on asset pricing of different ways on how to assign losses to principal and interest payments. When there is no default, interest payments can always be paid. In case of multi-period models, the assumption of identically and independently distributed returns is used.

The article *Reconsidering the appropriate discount rate for tax shield valuation* asks, as the title indicates, for a clarification of the appropriate discount rate for tax savings. With risk-free debt this is straight forward. Interest payments are risk-free so that tax savings are risk-free as well.³ Therefore, all tax savings are to be discounted at the risk-free rate. The respective literature already proposed some discount rates such as the risk-free rate, the yield rate⁴ or the cost of debt (compare for example Arzac and Glosten (2005) and Molnár and Nyborg (2013)). But questions remain unanswered. Especially, the assumptions under which the respective discount rates are to be used are not always clear. Even terms such as the cost of debt are not being used unambiguously.

In *Reconsidering the appropriate discount rate for tax shield valuation*, my co-author and I regard a setting without a taxation of cancelled debt. A default is triggered endogenously when, in any period, free cash flows and the levered firm value are not sufficient to fully repay contractual principal and interest payments. We basically use the discounted cash flow method. Discount factors are defined as conditional expected returns of the respective cash flows. Additionally, we take on a state-space view and use a stochastic discount factor (SDF)⁵ to price assets. This approach is more general than immediately applying the mean-variance CAPM, and it admits to draw some links to related research on option pricing such as the one by Coval and Shumway (2001). In the multi-period analysis, we only use single-period debt.

We decompose the full debt payments into principal payments and interest payments. Tax savings are just scaled interest payments in this setting. That is, when interest payments of all possible states are incorporated into the vector Int then tax savings are τInt , i.e., the interest payments scaled by the tax rate τ . We conclude that the discount rate for interest payments must be the same as the interest payments on tax savings, which is just an application of the law of one price.⁶

This discount rate on interest payments (and on tax savings) is equal to the discount rate on debt when the overall losses on debt are attributed proportionally to interest and principal payments. Proportional means here at the ratio of contractually agreed upon interest and principal payments. In cases without this proportional attribution, it is still possible that the discount rates

³This connection can be destroyed by earnings stripping rules or by settings in which insufficient profits are generated to pay the taxes in the first place. However, we do not consider those settings.

⁴For single-period coupon-paying debt the yield rate would be the coupon rate.

⁵The use of stochastic discount factor is relatively general in this paper. It could be the SDF of a single agent to obtain a valuation of the asset. It could also be the SDF within the payoff space.

⁶In a simple version the law says that a payoff X with a price $p(X)$ multiplied by a scalar a must have the price $ap(X)$, that is $p(aX) = ap(X)$. For example, with $a = 2$, earning double the payoff in any state must mean that the price of this payoff also doubles.

1.3. Description and integration of the different articles

are equal. However, this would be more of a coincidence, and one should expect that the discount rates regularly differ. We analyze the discount rates for two important priority assumptions.

When principal payments have priority, interest payments receive losses first. Only that part of losses greater than contractual interest payments is assigned to principal payments. Intuitively, this makes interest payments riskier than principal payments and also riskier than debt payments as a whole. Again intuitively, we expect the discount rate on interest payments to be higher than the one for principal payments. The discount rate for debt must be in between them because it is a weighted average of the discount rate on interest and principal payments, with nonnegative weights adding up to one.⁷ However, this intuition may not be correct. Interest payments can be represented as an option-like payoff with the overall debt payments as the underlying. Using this view and the results from Coval and Shumway (2001) on expected returns on option payoffs, we obtain conditions when the discount rate on interest payments is greater than, less than or equal to the one for debt.

The second priority assumption states that interest payments have priority over principal payments. In this case losses on debt are first assigned to principal payments. Again using intuition, that should make interest payments relatively safer versus principal payments. When interest payments are small relative to principal payments, a case that we see regularly in practice, interest payments may be close to risk-free in this case. That means the discount rate of interest payments should be less than the one of debt, which in turn should be less than the one of principal payments. But also for this case, to be more complete, we translate the results from Coval and Shumway (2001) for our purposes and obtain cases for which the before-mentioned relation does not hold.

In an illustrative example, we show discount rates of debt and interest payments (i.e., on tax savings) for different leverages. We restrict this example to the more intuitive and probably more likely cases. For interest prioritization much higher values for the tax savings can be obtained at the same leverage. This is because interest payments remain risk-free for a large range of leverages, but the coupon rate increases to reflect debt becoming riskier at higher and higher leverages. Thus, contractual interest payments increase but the discount rate for tax savings remains the risk-free rate or, at very high leverages, a bit higher than the risk-free rate. For principal prioritization interest payments become risky at much lower leverages than for interest prioritization. The discount rate on principal payments increases quickly with increasing leverage, so that the value of tax savings increases at a slower rate than for interest prioritization.

Turning to a multi-period analysis complicates several factors. One issue is to determine how the firm will behave after default. We make this analysis convenient through establishing a setting of independently and identically distributed returns. After a default, whoever is the new owner,

⁷The discount rate on debt discounts expected overall debt payments, which are the sum of principal and interest payments. From this one obtains that the discount rate on debt is a weighted average of the discount rates on interest and principal payments.

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will establish the same leverage as before. In this setting, at any point in time and in any state, the firms and its cash flows are just scaled versions so that the respective discount rates over any single period are equal. For this case, which is related to Miles and Ezzell (1985), we find that the original equation from Miles and Ezzell (1985), which was derived for risk-free debt, can also be used for risky debt.

The results have practical relevance. I have to admit that the analysis is stylized and ignores several issues that appear in practice, such as the inclusion of personal tax rates and the question of a more elaborate debt structure with debt differing in maturity, seniority, etc. This simplification is usually done, and with this article it is the same, to focus the analysis on a few important issues. Practitioners, such as valuers and financial analysts know that a practical implementation would be much messier. But they learn here that they need to know the regulations applicable in a default - especially how interest payments are treated. Standard textbook equations may implicitly assume a totally different setting and therefore bias the valuation. Depending on which rule is applicable, values of tax savings may be significantly higher or lower. Additionally, discounting tax savings at the risk-free rate, at the cost of debt or some other rate proposed may also bias the valuation when the underlying assumptions for such a choice are not matched.

This first article answers several questions regarding the discount rate for obtaining the value of tax savings. However, it also leaves a lot of questions open. Some of them are how a taxation of cancelled debt changes valuations, and how do actual valuation equations look like. My coauthor and I tackle those questions in the second article presented herein under the title *Valuation effects of taxes on debt cancellation*.

Our questions are how valuation equations differ when the assumption of a taxation of cancelled debt is used or not used. We also want to know under which of the two assumptions the equations for prices, tax-adjusted discount rates or WACCs are derived that appear in the respective literature but do not disclose which assumption is applied.

We restrict the analysis to a single-period setting. We derive valuation equations, adjusted discount rates, and textbook-like WACC equations for the case with and without cancelled debt. We keep the method simple in that we use a pricing operator to value cash flows. This way we do not have to apply a certain pricing model for valuation. Taking basic discount rates, such as the discount rate of unlevered equity and of debt as given, this is also not necessary. All valuation equations are derived relative to those basic discount rates.

A first important result is that when cancelled debt is taxed, valuation equations are independent from the distribution of losses on interest and principal payments. For example, both cases, a prioritization of interest and a prioritization of principal payments, lead to the same equation for the value and for the discount rate. Furthermore, using the same amount of debt, the value itself as well as the discount rate do not change using different assumptions on loss attribution. When cancelled debt is not taxed, the loss distribution matters. A simple equation, which also appears in the respective literature, can be derived for a pro rata (proportional) loss distribution of losses

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to interest and principal payments. In this case the coupon rate shows up as a parameter in the valuation or discount rate equation. We also discuss prior findings in the literatures such as Sick (1990), Arzac and Glosten (2005), Cooper and Nyborg (2008) and Molnár and Nyborg (2013). However, their models were not able to uncover the proportional loss feature or they were silent on what assumption on the taxation of cancelled debt is used.

For illustrative purposes we also provide an example. It shows how the application of an incorrect equation could bias the value of tax savings or the tax adjusted discount rate. The example assumes that interest prioritization is applicable, but instead the equations for a pro rata loss distribution are used. Differences in the tax shield value can go up to over 30% for very high leverages in this example.

The article adds some practical implications. It shows that when we ignore personal taxes, we still cannot rely on some simple equations that are regularly provided by the literature. When cancelled debt is always taxed, the situation is somehow diffused because there is one equation that fits all kinds of loss distributions. However, as soon as this is not the case, the loss distribution matters and needs to be known and to be reflected in the valuation equation.

The first two articles show that a lot can be done picking just a few assumptions. They can be extended in many ways, some of which I will describe more in detail in the conclusion. I decided to add an analysis of the beta de- and re-levering procedure. It seems to be a natural next step because it is very much related to discount rates.

I address the equations used for de- and re-levering in *De- and re-levering betas with risky debt revisited*. The procedure is important to obtain discount rates for a firm for which no or not enough pricing data is available. In this case the beta factor of a firm that is sufficiently equal may be used. Most of the time those firms use different leverages so that the beta needs to be de- and re-levered.

Here, I ask the question of how the de- and re-levering equations change for the different cases of risky debt with and without a taxation of cancelled debt. I want to know what kind of parameters need to be considered, and I want to get an intuition on how much of an impact the application of an incorrect procedure would have.

The core part is a single-period analysis. The expected return equation of the mean-variance CAPM provides the basis for the derivation of beta factors. To start with low complexity and to show the basic method, I rederive the delevering equation for risk-free debt. Then, I turn to risky debt and go through the different cases. When cancelled debt is taxed, only one equation is needed that covers all loss distributions. If that is not the case, one has to look at the type of loss distribution. I treat proportional loss as well as interest and principal prioritization.

The resulting beta equations suggest that with risky debt usually only the case with a taxation of cancelled debt is considered. As I point out in the beginning of the article, there are many exceptions to a taxation of cancelled debt for example in the United States. Thus, it is not advisable to ignore it. If cancelled debt is not taxed, different equations have to be used for

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different distributions of losses on interest and principal payments. For interest prioritization the assumption of risk-free interest payments makes sense. This assumption leads to a simple equation. However, for principal payments one can also use the assumption of risk-free principal payments, but this is a case that is highly unlikely in practice. For not to refer to more complex mathematics, I introduce an additional parameter that marks the difference of the price of losses for principal prioritization versus the one for pro rata prioritization. An illustrative example gives some intuition on the impact of the different terms in the beta equations. The example suggests that it matters a lot whether debt is risk-free or risky. Ignoring this and assuming risk-free debt may strongly bias the de-levered beta. It also matters whether cancelled debt is taxed or not. It matters much less in the example whether losses are distributed pro rata, or according to interest or principal prioritization. Unlevered betas do not differ a lot when switching from one to another equation. A note of caution here, even though the parameterization in the example is realistic, it is just a single example and not a sensitivity analysis. Thus, it does not claim general validity.

This third article moves along the lines of company and tax shield valuation. I do not want to limit my research interest on single-firm or single-asset valuation. For this reason, I have extended my work towards some more macroeconomic settings in which agents solve an optimization problem and markets clear. In the context of asset pricing and taxes the article *Effects of a capital gains tax on asset pricing* continues and extends the work of Kruschwitz and Löffler (2009).

As in Kruschwitz and Löffler (2009) I ask for conditions when two endowment economies, with the same initial conditions and with the only difference in the existence of a capital gains tax in one economy (called tax economy), exist, that have the same prices or price vector in equilibrium. However, I ask this question for a setting in which agents maximize expected utility over consumption, whereas Kruschwitz and Löffler (2009) look at a mean-variance CAPM.

I write down the agents' problem, which is the maximization of expected utility over consumption given their initial endowment with assets. Investors' utility functions, their endowments as well as assets and their payoffs define the initial situation of the economy. The tax economy only adds a capital gains tax on all asset returns. Taxes are transferred back to the investors according to some predefined rule. Thus, taxes remain in the economy and are not treated as "burned money" or wasted resources. The tax rate is certain and equal for all assets. The taxation is also symmetric in that capital gains lead to taxes and losses to a tax return. I derive pricing equations, which express prices as expected cashflows that are discounted by a stochastic discount factor. I use those pricing equations to derive conditions for price equality. I go on to look at a model with consumption at two times and in which the risk-free rate is zero as well as a model with consumption only at time one.

I find that a sufficient condition for equal prices in the two economies is a zero risk-free rate. I show that when the consumptions of every single agent in every state is the same in both economies, the stochastic discount factors of all agents are the same and prices as well. The portfolio rule that guarantees equal consumption in this expected-utility-over-consumption setting is the same

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as the one in Kruschwitz and Löffler (2009). Capital gains taxes have a zero price with a zero risk-free rate. When the risk-free rate is not zero, one may not be able to find the tax and the no-tax economy in equilibrium with equal prices. When marginal utility is linear, i.e., agents have quadratic utility, prices are never equal in the tax and the no-tax economy.

Kruschwitz and Löffler (2009) actually only look at a CAPM with consumption at time one and without current consumption. I continue with this type of economies. In this case the risk-free rate is exogenous, i.e., it can be picked. With a zero risk-free rate, equal price vectors can be obtained again. When the risk-free rate is not zero, I find two cases for which equal prices can be obtained. The first case is exponential utility with multivariate normal payoffs. Even with equal price vectors, aggregate wealth is different in the tax and the no-tax economy. This must be true because in the tax economy aggregate tax payments (or transfer payments) have now positive value and add to the aggregate prices of the assets. I obtain the same portfolio rule for risky assets as for the case with the zero risk-free rate. Only the rule for the risk-free asset differs. The second case for which equal prices can be obtained is when all investors have linear marginal utility. In this case the stochastic discount factors of agents are equal in equilibrium. Due to linearity of the SDFs all of them are in the payoff space, which is also the consumption space. Since there can be only one SDF in the payoff space, all of them must be equal. They are also equal to the aggregate SDF, which is just a scaled version of aggregate consumption and aggregate bliss point consumptions. This way after-tax payoffs in the tax economy and untaxed payoffs in the no-tax economy are discounted to obtain the same prices. Other types of utility functions do not necessarily produce SDFs within the payoff space so that one cannot generally find tax and no-tax economies with equal prices.

I continue to describe the limitations of the analysis. One of them is the symmetry of the taxation, which we do not see in practice. Usually, gains are taxed but losses can only reduce gains in the period they both appear or, if losses are greater than gains, losses may be carried forward for some time. This feature would destroy the observation that the price of a capital gains tax is zero with a zero risk-free rate. Thus, the article is more a model-theoretic work. The practical implications remain limited. At the lower end we know that capital gains taxes may well distort prices versus an economy without taxes. For a modeler those insights may be useful for not only picking special cases for a further analysis of a capital gains taxation that may lead to convenient results but give a biased analysis.

This paper closes my analysis on tax effects on asset prices. I am completely aware that they leave many issues unanswered. I address those that I find most pressing in the last chapter. There, I give an outlook on how research can be continued in the area of taxes and asset pricing.

In the subsequent four chapters, I will present the four different articles in the order presented in Table 1.1. Versus the published version, I added a paragraph in front of each article to provide some context and background. As mentioned before, I summarize my work and give an outlook for possible continuing research for the topics at hand in the final chapter.

Chapter 2

Reconsidering the appropriate discount rate for tax shield valuation¹

¹This chapter is based on Krause and Lahmann (2016). Reprinted by permission from Springer Customer Service Centre GmbH: Springer Nature ZEITSCHRIFT FÜR BETRIEBSWIRTSCHAFT (Reconsidering the appropriate discount rate for tax shield valuation, Marko Volker Krause and Alexander Lahmann), ©Springer-Verlag Berlin Heidelberg 2015, (2016).

Bringing this publication into context

This article is chronologically, but also logically the first academic contribution of this dissertation. The paper rests on the insight that corporate distributions go to equity holders, debtholders (if debt is taken out) and to the tax authorities. All these distributions are valuable so that the asset value of a firm is the sum of the value those distributions. Furthermore, the corporate finance literature usually takes on the view of equity- and debtholders, i.e., an investor perspective so that firm value is regularly defined as the sum of the debt and the equity value, without considering the value of tax payments. However, there is a way to decrease taxes paid to the tax authority and, therefore, to increase firm value. By taking out debt, interest payments are due. In the tax jurisdictions we regard, interest payments must be deducted from pre-tax profits so that less corporate taxes are paid, i.e., tax savings are incurred. Those tax savings are the main object of this paper. Especially, we look at the discount rate of corporate tax savings, often also called the tax shield. We define discount rates as the conditional expected return of the respective payoff.

Tax shield valuation is far from being a new thing. However, little has been done for defaultable debt. We take on this case and look at the discount rate of tax savings under different assumptions of how losses are assigned to interest and principal payments. When interest payments are prioritized, losses are first assigned to principal payments. That keeps interest payments relatively safe. Assuming that also in default tax savings can be incurred, the firm may still enjoy tax savings on the interest payments made. Due to the relative safety of interest payments versus principal payments and versus the overall debt payments, the discount rate for interest payments is regularly not equal to the one for debt. We conclude the following:

In our setting, tax savings are proportional to interest payments. Therefore, tax savings have the same discount rate as interest payments. Furthermore, only for a pro-rata distribution of losses between interest and principal payments the discount rates of interest payments, tax savings, principal payments and the overall debt issue are always equal. For other distributions they may be equal but are regularly not. A numerical example for interest and principal prioritization gives an idea of how big differences between discount rates for debt and for tax savings can be. We also provide some qualitative statements on the relations between discount rates of interest payments, tax savings, principal payments, and debt payments as a whole. Thereby we draw from conclusions from option pricing. Finally, we show that, under the usual and very simplifying assumptions in the corporate finance literature, we can implement the findings into multi-period valuation equations.

This paper marks the starting point for the papers that follow. Naturally, a paper can properly analyze only a limited amount of issues. That is why we restricted it to a certain set of assumptions such as assuming that cancelled debt is not taxed. In the next chapter we explicitly look at this assumption.

Abstract

This paper aims at identifying the appropriate discount rate for tax shield valuation in a setting where a partial default is possible and either principal or interest payments are prioritized in default. As a general valuation framework we use the stochastic discount factor. We assume a tax framework with corporate taxes, tax-deductible interest payments of the firm, no taxes on the cancellation of debt and no personal taxes. We strictly decompose the payments owed to the debtholders into interest and principal payments and analyze discount rates of those claims for the different priorities. As a result of the single-period analysis we find that the discount rate for tax savings, i.e., the conditional expected return on tax savings, is always equal to the discount rate of debt only for a proportional loss distribution on interest and principal payments. If losses are distributed according to one of the priority assumptions, the discount rate of tax savings behaves different from the discount rate of debt and both discount rates are equal only in very special cases. Furthermore, we derive qualitative statements for the relation between the discount rate of debt and the discount rate of tax savings assuming certain correlations between the stochastic discount factor and the debt repayments. Finally, we show how the prioritization assumptions can be implemented in a multi-period setting. We obtain for the presented set of assumptions a pricing equation equivalent to the one by Miles and Ezzell (1985).

Keywords: Tax shield, Firm valuation, Default Risk, Stochastic Discount Factor

2.1 Introduction

Since the seminal contribution of Modigliani and Miller (1958) and (1963), the fact that interest payments on debt reduce the tax base, and firms who partially finance their operations with debt thereby save taxes, is one of the most prominent results in corporate finance. As a logic consequence Myers (1974) proposed the direct valuation of these tax savings, the so-called tax shield, as a separate term in the adjusted present value (APV) approach. The appropriate pricing of these tax savings heavily depends on the considered setting and is subject to an ongoing academic discussion. Primarily, the academic debate centered around two major issues: First, the literature stream which originated from the influencing work by Miles and Ezzell (1980) and (1985) and deals with the influence of the considered financing policy upon the tax shield (see e.g. Fernandez (2004), Fieten et al. (2005), Cooper and Nyborg (2006), Massari et al. (2007) and recently Dempsey (2013)); and second, the literature stream considering the impact of default risk (see e.g. Homburg et al. (2004), Kruschwitz et al. (2005), Rapp (2006), Cooper and Nyborg (2006), Kruschwitz and Löffler (2006), Molnár and Nyborg (2013) and recently Koziol (2014)).² The present study belongs to this latter literature stream and specifically deals with the impact of a partial default. In case of a partial default, the debtor might be able to pay a part of the contractually fixed payments owed to the debtholders. Depending on the legal framework, the debt contract or the decision of the involved parties, the partial payments are prioritized to pay down either principal or interest first. In this article, we discuss the impact of this prioritization on the risk structure (i.e., riskiness) and the valuation of the tax shield.

The literature stream already clarified that the standard discount rate for the tax savings, assuming a constant leverage and risk-free debt, is the risk-free rate for one period and the discount rate of unlevered equity for all remaining periods (see e.g. Miles and Ezzell (1985)). For risky debt, Arzac and Glosten (2005) suggest the expected return on debt, i.e., the cost of debt, instead of the risk-free rate.³ Several other articles discussed the inclusion of risky debt in firm valuation in more detail. For example, Kruschwitz et al. (2005) showed under specific assumptions, e.g. the taxation of cancellation of indebtedness, that default risk has no impact on firm pricing. In contrast to this, Rapp (2006) starts from the premise of an exemption of a taxation of a cancellation of indebtedness and demonstrates that ignoring the risk of default might lead to a significant valuation bias. Several studies such as Cooper and Nyborg (2008) or Couch et al. (2012) prove this effect. In addition, there is a slightly different literature stream on identifying conditions for default neutral tax systems (e.g. Bond and Devereux (2003) or Blaufus and Hundsdorfer (2008)). As a matter of

²This impartial list only contains articles dealing with the discounted cash flow (DCF) approach. Other articles on the optimal capital structure implicitly consider the effect of default on the tax shield (see e.g. Leland (1994) or Goldstein et al. (2001)).

³While it is still under discussion whether Miles and Ezzell (1980) considered risky debt (see e.g. Rapp (2006), p. 777, footnote 24), they refer to the term “cost of debt” in their derivations. Koziol (2014), p. 656, uses the cost of debt for a default-risk adjusted WACC approach. With a respective rearranging the cost of debt would then represent a part of the discount rate of the tax shield.

2.1. Introduction

fact, this literature stream touches several tax topics which are relevant for quantifying the effect of default risk upon the tax shield. More closely related to the present study, only few works, such as Molnár and Nyborg (2013), considered the impact of partial default and the prioritization of interest or principal payments. In a simplified two-state framework they derive equations for tax-adjusted discount rates. In their case, the fraction of debt that is lost in default has to be known upfront, i.e., it is not a random variable but a given parameter. Besides Molnár and Nyborg (2013), Swoboda and Zechner (1995) also introduce the prioritization of interest or principal payments but do not discuss their effect on discount rates.

Our paper aims at identifying the risk structure of the tax savings and deriving the appropriate discount rate based upon the stochastic discount factor (SDF) approach. By acknowledging that the interest payments are the main driver of the tax savings, we draw upon relevant assumptions with respect to the default regime such as the prioritization of interest or principal, the tax deductibility of interest in the event of default or the taxation of a possible cancellation of indebtedness. Moreover, we extend the results of the literature dealing with tax shield valuation considering default by deriving qualitative statements for the relation between the discount rate of the firm's debt issue and the tax savings using the results of Coval and Shumway (2001). In order to reduce complexity, we first discuss the effects of different priority assumptions in a single-period model and then extend our analysis towards a multi-period setting. Since we regard a classical (Modigliani-Miller-like) model framework, we do not consider personal taxes. On the one hand, this enables us to focus on the effects on the firm level, on the other hand, however, it does not regard the possible effects of a credit default on the level of debt- and equityholders.

We contribute to the literature stream on pricing tax savings in several ways. As a main result, we find tax savings to have a different risk structure than the overall debt issue. Therefore, we strictly differentiate between the overall debt issue, the interest and the principal payments. In contrast to other studies⁴ we demonstrate the differences between the risk structures of these debt obligations and show for which case they have a risk equivalent to the one of the overall debt issue. Moreover, we show the risk equality between the tax savings and the interest payments. An important consequence is that both should have the same discount rate. Additionally, we derive qualitative statements of the relation of the discount rate of tax savings and of debt for the different prioritization assumptions.

This article is organized as follows: In section 2.2 we briefly introduce the basic terms of the applied SDF approach and firm pricing. A single-period analysis is conducted in section 2.3. Section 2.4 focuses on the multi-period setting and shows the inclusion of our findings in a typical Miles-Ezzell framework. Section 2.5 concludes.

⁴Examples are Molnár and Nyborg (2013) or Koziol (2014).

2.2 Definition of basic terms

2.2.1 The SDF and its interpretation

Throughout our analysis we use the stochastic discount factor (SDF) approach for pricing the firm subject to default risk. The SDF is not very often applied for firm valuation.⁵ Nevertheless, we utilize this approach since the SDF is a very general approach and, by adding few specific assumptions, enables us to make economic interpretations of the pricing procedure. Therefore, we introduce the necessary assumptions, show the basic pricing equations and discuss their implications.⁶

For the definition of the basic terms of the SDF it suffices to regard two points in time t and $t + 1$. We use the conditional expectations operator $E_t[\cdot]$, where for example $E_t[x_{t+1}]$ indicates the conditional expected value of a discrete N -state random variable x_{t+1} conditional on the available information at time t . The state-specific realization is denoted by $x_{t+1}(\omega)$, where the index $\omega = 1, \dots, N$ describes the state.

The basic SDF-pricing equation is given by

$$p_t(x_{t+1}) = E_t[m_{t+1}x_{t+1}], \quad (2.2.1)$$

where $p_t(x_{t+1})$ denotes the price at time t of the random cash flow x_{t+1} and m_{t+1} the investor's random discount factor, i.e., the SDF. We assume an arbitrage-free (capital) market, which implies a strictly positive SDF, i.e., $m_{t+1} > 0$. In order to be able to state an economic interpretation we assume risk-averse investors, who maximize utility over consumption so that the SDF is marginal utility growth of consumption. Risk-aversion (a concave utility function) and marginal utility defined over consumption imply a negative correlation between an investor's SDF and its individual consumption. Such an investor values cash flows appearing in bad states (low consumption and high marginal utility growth) higher than cash flows appearing in states with already high consumption (low marginal utility growth).⁷

We define the gross return of the uncertain cash flow x_{t+1} over the period t to $t + 1$ by

$$R_{t+1} = \frac{x_{t+1}}{p_t(x_{t+1})}, \quad (2.2.2)$$

and net returns as $r_{t+1} = R_{t+1} - 1$. A certain cash flow of one unit arriving at $t + 1$ has to return the gross risk-free rate R_t^f .⁸ We get the relation $p_t = \frac{1}{R_t^f} = E_t[m_{t+1}1] \Leftrightarrow R_t^f = \frac{1}{E_t[m_{t+1}]}$. By

⁵One notable exception is Arzac and Glosten (2005).

⁶For further expositions and derivations we refer to Cochrane (2005).

⁷We abstain from further assumptions with regard to the economy. However, let us briefly line out some additional remarks: For the case of complete markets, all investors have the same unique SDF, there is perfect risk sharing and the investors' individual consumption moves in lockstep with aggregate consumption, i.e., moves in lockstep with business cycles. For the case of incomplete markets, there can be many SDFs but there is a unique SDF within the payoff space of tradable assets, the projection of every investor's SDF onto the payoff space of tradable assets (see for example Cochrane (2005)). In this case, one might be faced with imperfect risk sharing and individual consumption might not move in lockstep with aggregate consumption.

⁸Since the risk-free rate is known at time t we use t as subscript as opposed to uncertain returns.

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taking the conditional expectations of equation 2.2.2 we are able to restate this relation to

$$E_t[R_{t+1}] = \frac{E_t[x_{t+1}]}{p_t(x_{t+1})}, \quad (2.2.3)$$

and define single-period conditional expected returns as discount rates.⁹ An alternative representation of conditional expected returns which is applied throughout our analysis is given by

$$E_t[R_{t+1}] = R_t^f [1 - cov_t(R_{t+1}, m_{t+1})]. \quad (2.2.4)$$

Equation 2.2.4¹⁰ shows that conditional expected returns decrease with increasing covariance of returns with the SDF, i.e., with marginal utility growth of consumption. Thus, investors value assets higher (higher price and lower conditional expected return) that are more likely to have relatively high cash flows in states where their consumptions is low and low cash flows in states where their consumption is already high. The more negative the covariance between returns and the SDF is, the higher in turn are the conditional expected returns and the lower the prices. These economic interpretations are useful for the remainder of the analysis and can be regarded as the main reason for the choice of this pricing concept.¹¹

2.2.2 Basic considerations for firm pricing

In our analysis we differentiate between discrete points in time $s = t, t + 1, t + 2, \dots, T$. We regard a levered firm with limited liability, i.e., the equity value is greater than or equal to zero at any time. We denote the value of the levered firm by V_s^L . At an arbitrary time s the operations of the firm generate an uncertain unlevered free cash flow stream FCF_s^U , where its state-dependent value is given by $FCF_s^U(\omega)$, with $\omega = 1, \dots, N$ and $0 \leq FCF_s^U(1) < FCF_s^U(2) < \dots < FCF_s^U(N)$. Additionally, we require the unlevered free cash flows and the investment program of the firm to be independent of its financing activities, i.e., of the leverage l , with $l \in [0, 1)$. This implies that we work in a Modigliani-Miller-like framework.¹² At an arbitrary time s the value of the outstanding total amount of debt is denoted by D_s . The firm pays interest Int_s and net principal payments by $PP_s^{net} = PP_s - D_s$ ($PP_s^{net} < 0$ indicates a (net) issuance of debt and $PP_s^{net} > 0$ a decrease

⁹Several authors define discount rates as cost of capital (see Kruschwitz and Löffler (2006), Laitenberger and Löffler (2006) or Cooper and Nyborg (2008)). In this literature stream cost of capital are defined as conditional expected one-period returns with additional assumptions such as cost of capital being deterministic (see Kruschwitz and Löffler (2006)). We keep the term expected return throughout our analysis and add assumptions when necessary.

¹⁰To derive equation 2.2.4 we use the price equation 2.2.1 and expand it to $p_t(x_{t+1}) = E_t[m_{t+1}]E_t[x_{t+1}] + cov_t(m_{t+1}, x_{t+1})$. Now, we divide by the price $p_t(x_{t+1})$, substitute in $\frac{1}{R_t^f}$ for $E_t[m_{t+1}]$ to obtain $1 = \frac{E_t[R_{t+1}]}{R_t^f} + cov_t(m_{t+1}, R_{t+1})$. Multiplying by R_t^f and rearranging for $E_t[R_{t+1}]$ we arrive at equation 2.2.4.

¹¹Alternatively, we could have applied the risk-neutral pricing approach, which became famous especially through its application in derivative pricing. If applied correctly and stating the necessary assumptions, i.e., in this study an arbitrage-free (capital) market and additionally a complete market, risk-neutral and SDF pricing yield equivalent results. In this case, the existence of a unique risk-neutral probability measure implies a positive SDF. However, as stated above and more specifically in footnote (FN) 7, our pricing arguments hold without the additional assumption of a complete market.

¹²See Kim et al. (1993), p. 119f., or Kruschwitz and Löffler (2005), p. 223f., for a discussion of these assumptions.

2.3. Risky debt with partial default - a single-period analysis

(repayments) of debt), where PP_s denotes the principal payments of debt issued at time $s - 1$.¹³

The market value of a levered firm V_t^L conditional on the available information at t can be determined by using the adjusted present value (APV) approach (see Myers (1974)) according to

$$V_t^L = V_t^U + VTS_t, \quad (2.2.5)$$

where V_t^U is the value of the otherwise identical but unlevered firm and VTS_t the present value of all future tax savings. Note that V_t^U is determined via the SDF by $V_t^U = \sum_{s=t+1}^T E_t[m_s FCF_s^U]$, where $T \rightarrow \infty$ is possible (see Arzac and Glosten (2005)). The unlevered free cash flows are defined in the well-known way $FCF_s^U = EBIT_s(1 - \tau) - Inv_s$, where $EBIT_s$ denotes the earnings before interest and taxes and Inv_s the net investments. For simplification purposes we set the net investments (investments minus depreciation) to zero. The corporate tax rate τ ¹⁴ is assumed to be constant. Applied to the $EBIT_s$, we obtain the tax payments of an unlevered firm $Tax_s^U = \tau EBIT_s$.

We define tax savings TS_s of a period s as the difference between the taxes of an unlevered and a levered firm, i.e., $TS_s = Tax_s^U - Tax_s^L$, so that the value of the tax savings is determined by (see e.g. Arzac and Glosten (2005), p. 454)

$$VTS_t = \sum_{s=t+1}^T E_t[m_s TS_s] = \sum_{s=t+1}^T E_t[m_s Tax_s^U] - \sum_{s=t+1}^T E_t[m_s Tax_s^L], \quad (2.2.6)$$

where the taxes of a levered firm Tax_s^L depend on the corporate tax rate τ ¹⁵, the EBIT and effects related to tax payments such as debt related tax savings. For risky debt, the explicit definition of Tax_s^L depends on several assumptions that we discuss in section 2.3.1.

2.3 Risky debt with partial default - a single-period analysis

In this section, we derive expressions for the discount rate of tax savings, i.e., the conditional expected returns on tax savings, and qualitative statements about its relation to the discount rate of debt. Within our set of assumptions, interest payments are the core variables that determine tax savings. We expose how the prioritization of principal over interest or interest over principal payments affects the risk of tax savings when a (partial) default is possible. To reduce complexity, we regard a simple single-period model where the firm exists from t to $t + 1$. At t the firm issues

¹³Throughout our analysis we regard single-period debt contracts, i.e., the amount of debt issued at time $s - 1$ is promised to be repaid at s .

¹⁴By excluding an explicit modelling of personal taxes we operate - from a tax perspective - in a simplified framework, which is predominant in the corporate finance literature stream dealing with the tax benefits of debt financing (firm valuation literature see e.g. Rapp (2006), Koziol (2014); capital structure theory Titman and Tsyplakov (2007) or Hackbarth and Mauer (2012)). Nevertheless, we discuss at the respective points of our analysis the consequences.

¹⁵At this point, disregarding the effects of default, we already can outline one consequence of not considering personal taxes. While interest payments on the corporate level help to avoid corporate income tax, (1.) interest income on the personal level has to be taxed, implying a decrease of the tax savings from perspective of the personal level and (2.) payments to equityholders are taxed on the personal level as well. Here, it is relevant to note that interest payments reduce the payments to equityholders, which results in lower personal equity taxes. See for a detailed analysis of the US e.g. Graham (2003).

2.3. Risky debt with partial default - a single-period analysis

an amount of debt D_t . At $t + 1$ the firm's operations generate an uncertain unlevered free cash flow FCF_{t+1}^U , and interest Int_{t+1} as well as principal PP_{t+1} are paid on the total amount of debt D_t .¹⁶ Depending on whether default occurred or not and the subsequently discussed consequences of default the firm might realize tax savings. Note that this single-period model has an important consequence: The levered firm value amounts at $t + 1$ in each state to $V_{t+1}^L = 0$ ¹⁷. Note that some of our subsequently discussed assumptions are valid for the multi-period case as well.

2.3.1 Assumptions and discussion

We emphasize the possibility of a partial default, in opposition to models where default strictly implies a full loss.¹⁸ Therefore, we have to define the states of no default and (partial) default. Without a default the firm fully distributes the promised principal and interest payments $D_t(1+r_t^c)$ to the debtholders. Here, r_t^c is the promised yield for the period t to $t + 1$.¹⁹ In default the firm is only able to pay an amount smaller than the promised interest and principal payments $D_t(1+r_t^c)$, whereas a partial default means that the amount the firm can pay to debtholders is greater zero. Without having defined the available cash flows in default yet, we may already state some conditions regarding the difference between the promised as well the actually paid interest and principal payments, which we define as loss L_{t+1} of the promised debt repayments. Since we regard a limited liability firm and consider a model framework without possible bankruptcy costs²⁰, the losses in default maximally amount to the contractually fixed debt payments $L_{t+1} = D_t(1+r_t^c)$, i.e., the firm is unable to pay anything to the debtholders. The minimum loss coincides with the no default case, i.e., the promised interest and principal payments are properly rendered, which implies $L_{t+1} = 0$. Since in a model framework allowing for partial default we regard, besides the aforementioned extreme scenarios, all cases in between, i.e., the firm may not completely default on its debt obligations, the loss in any state ω at time $t + 1$ is defined by $L_{t+1}(\omega) \in [0, D_t(1+r_t^c)]$. We denote the actual repayment distributed to debtholders as RP_{t+1} , which is the sum of actually paid principal and interest payments, i.e., $RP_{t+1} = Int_{t+1} + PP_{t+1}$, or in notation with contractual payments and losses to debt holders $RP_{t+1} = D_t(1+r_t^c) - L_{t+1}$.²¹ The assignment of the losses to interest and principal payments depends on their respective prioritization. From an economic perspective it might be useful for the firm to pay interest in order to achieve tax savings on interest payments in the period where the default occurs. However, most tax codes claim (see e.g. Swoboda and Zechner (1995), p. 781) that principal payments have priority over interest payments (Main

¹⁶Note that PP_{t+1} and PP_{t+1}^{net} coincide in the single-period analysis.

¹⁷The perspective is here from the moment in $t + 1$ when the cash flow has just been paid out.

¹⁸Partial default has been analyzed by Molnár and Nyborg (2013) in a binomial model where the loss in default is ex ante specified. In comparison, we regard a model with more than two states and without relying on the assumption of a deterministic loss.

¹⁹We use the subscript t for the period from t to $t + 1$ because the promised yield is already known, i.e., determined at t .

²⁰Bankruptcy costs could easily be implemented as a factor reducing the firm value in case of default (see e.g. Koziol (2014)).

²¹Note that in the single-period case principal and net principal payments are the same, because in $t + 1$ the firm ceases to exist.

2.3. Risky debt with partial default - a single-period analysis

assumption 1.1 below). Only some country-specific laws, such as the German civil code (§366 BGB), allow that the debtor chooses how debt payments should be prioritized. Then the debtor might choose to prioritize interest payments in order to incur the maximum amount of possible tax savings (Main assumption 1.2). Note that only one prioritization rule can be applied at a time. Therefore, we split the respective analysis into two sections 2.3.3.1 and 2.3.3.2 where either main assumption 1.1 or 1.2 is respectively presupposed.

Main assumption 1 (Prioritization of interest payments):

The prioritization of interest payments over principal in the event of default depends on a respective covenant or the decision of the levered firm.

Main assumption 1.1 (A1): Principal payments are prioritized over interest payments.

Main assumption 1.2 (A2): Interest payments are prioritized over principal payments.

We continue to define the tax payments of a levered firm with a possible (partial) default. In case of a default, the tax deductibility of the actually paid interest depends on whether the tax authority allows their deductibility.²² Here, we continue on the assumption that, in the event of default, the tax authority allows the tax deductibility of the actually paid interest.²³

Main assumption 2 (Tax deductability of interest payments):

The tax authority always grants the tax deductibility of the actually paid interest.

As outlined by Cooper and Nyborg (2008), among others, a crucial assumption with respect to the impact of default on the firm value is the respective treatment of a possible cancellation of indebtedness (COD) by the tax authority. The fact that the levered firm has to write down some of its debt might give rise to a tax liability, i.e., a tax on a COD. The treatment depends on the jurisdiction's tax code.²⁴ For example, in the United States, the COD gives rise to a tax liability (IRC Section 61(a)(12)), but an exception for several reasons is possible, such as for a reorganization under Chapter 11 (see Miller (1991) or for more details USC §108). Other countries belonging to the G7 allow a tax-free COD as well, e.g. the German tax legislation allows for a tax-free COD supporting the continuance of the firm's operations (see §227 AO and BFH decision

²²Blaufus and Hundsdorfer (2008) discuss the impact upon the paid taxes of whether the tax authority allows for a tax deductibility of interest payments or not.

²³Kruschwitz and Löffler (2005), p. 228, proceed on an equivalent assumption in combination with a strict prioritization of interest payments. With their assumption of a minimum cash flow for at least paying the taxes this results - even in default - in a constant tax deductibility of the contractually fixed interest payments. In Rapp (2006), specifically p. 777, the therein stated assumptions imply that the firm's contractually fixed interest payments are always tax-deductible.

²⁴Cooper and Nyborg (2008) elaborate that one of the differences with the highest impact on tax shield valuation between Sick (1990) and Miles and Ezzell (1980) is the different assumption with respect to the tax treatment of a COD. This difference has been further analyzed by Blaufus and Hundsdorfer (2008) by including personal taxes. They show that the main difference between the aforementioned approaches as well as the ones by Kruschwitz and Löffler (2005) on the one hand and on the other by Homburg et al. (2004) and Rapp (2006) result from the general tax treatment of a default, i.e., the tax deductibility of interest payments, the taxation of a possible COD and the taxation of a possible debt write-down on the level of the debtholder.

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as of June 14, 2010 - X R 34/08). As a direct consequence, it is a valid simplifying assumption²⁵ that an exception of the taxation on a COD is granted.

Main assumption 3 (Taxes on cancellation of indebtedness (COD)):

The amount of debt L_{t+1}^{PPP} that is cancelled at an arbitrary time $t + 1$ will not be taxed by the tax authorities.

Main assumption 3 together with our exclusion of taxes on the debtholder and shareholder level combined with main assumption 2 imply that the period-specific tax savings at an arbitrary period time point $t + 1$ are always determined by $TS_{t+1} = \tau Int_{t+1}$, i.e., the difference between Tax_{t+1}^U and Tax_{t+1}^L .²⁶ According to our set of assumptions Tax_{t+1}^L is given by $Tax_{t+1}^L = (EBIT_{t+1} - Int_{t+1})\tau$.²⁷ In case of a COD taxation, the period-specific tax savings have to include the respective taxation and would therefore amount to $TS_{t+1} = \tau Int_{t+1} - \tau L_{t+1}^{PPP}$.²⁸ The aforementioned assumptions imply that the levered free cash flows can be generally determined at an arbitrary time point $t + 1$ by

$$FCF_{t+1}^L = FCF_{t+1}^U + TS_{t+1} = FCF_{t+1}^U + \tau Int_{t+1}. \quad (2.3.1)$$

After having discussed the assumptions for a partial default we are well equipped for obtaining an expression quantifying the loss L_{t+1} in the single-period setting. Since even in default taxes are saved at τInt_{t+1} , the debtholders obtain the unlevered free cash flows plus possible tax savings, i.e., the levered free cash flows considering default FCF_{t+1}^L . Independent of the prioritization of payments the losses on the promised debt payments are then defined as the positive difference between the promised interest and principal payments and the levered free cash flows:

$$L_{t+1} = \max(0, D_t(1 + r_t^c) - FCF_{t+1}^L). \quad (2.3.2)$$

²⁵As stated above, assuming no tax on COD is a valid assumption for mapping the case of a reorganization (see e.g. Miller (1991) or Cooper and Nyborg (2008)). Therefore, at first glance, it might not be reasonable to combine a single period model with the assumption of no taxes on a COD. In a single period setting, the firm is simply not able to reorganize. However, we have decided to state this assumption with regard to the more general setting of the multi-period analysis in section 2.4 which aims at mapping the valuation of a firm that uses in default the opportunity to reorganize. Notice that no tax on a possible COD is an implicit standard assumption in most works on optimal capital structure (e.g. Leland (1994)), even though most works dealing with optimal capital structure do not consider the case of a reorganization.

²⁶We consider a strict prioritization of the payments to the tax authority (equivalently Kruschwitz and Löffler (2005)). Thereby, the firm cannot default on its tax payments. With our set of assumptions, we always compare the unlevered free cash flows with the debt repayments where taxes have been already deducted.

²⁷Typically, Tax_{t+1}^L is defined in terms of the equity cash flows (ECF_{t+1}), with our set of assumptions, by $Tax_{t+1}^L = (ECF_{t+1} + PP_{t+1}^{net}) \frac{\tau}{1-\tau}$, where $ECF_{t+1} = (EBIT_{t+1} - Int_{t+1})(1 - \tau) - PP_{t+1}^{net}$.

²⁸Assuming a taxation of a COD implies a kind of irrelevance of default on the value of tax payments. This tax effect has already been shown by Sick (1990) and Kruschwitz et al. (2005). Blaufus and Hundsdorfer (2008), p. 173, show the effects of such a tax system and draw upon the taxation effect of debt write-downs on the creditor level. Combining the case of tax deductibility of interest payments on the corporate level with a taxation of a COD and the tax deductibility of debt write-downs on the creditor level results in a tax system which does not distort the effects of a default (credit default neutral tax system). For the assumptions of Homburg et al. (2004) and Rapp (2006), as already discussed in FN 24, Blaufus and Hundsdorfer (2008) analyze how the tax treatment of debt write-downs on the creditor level has to be constructed in order to imply a credit default neutral tax system.

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2.3.2 General properties of the single-period model with risky debt

After having discussed the consequences and the tax treatment of default, we turn our attention towards the calculus of the debtholders to derive a relation for the promised yield r_t^c , which is contractually fixed at t . Moreover, since we are interested in the appropriate discount rate for the tax savings we define and analyze the returns for the different debt cash flows - interest and principal - as well as for the tax savings. We start with the promised yield by representing the value of debt as discounted promised interest and principal payments subtracted by the loss on debt repayments:

$$\begin{aligned} D_t &= E_t[m_{t+1}(D_t(1 + r_t^c) - L_{t+1})] \\ &= \frac{D_t(1 + r_t^c)}{R_t^f} - E_t[m_{t+1}L_{t+1}], \end{aligned} \quad (2.3.3)$$

where the second equality uses $R_t^f = 1/E_t[m_{t+1}]$. We rearrange for the promised yield r_t^c and obtain

$$r_t^c = r_t^f + \frac{E_t[m_{t+1}L_{t+1}]R_t^f}{D_t}. \quad (2.3.4)$$

As a short-hand notation we define the loss ratio $\phi_{t+1} = \frac{L_{t+1}}{D_t}$ and restate equation 2.3.4 to

$$\begin{aligned} r_t^c &= r_t^f + E_t[m_{t+1}\phi_{t+1}]R_t^f \\ &= r_t^f + cov_t[m_{t+1}, \phi_{t+1}]R_t^f + E_t[\phi_{t+1}]. \end{aligned} \quad (2.3.5)$$

This equation shows several intuitive results: In the case of risk-free debt, the loss ratio is zero in any state so that $r_t^c = r_t^f$. In the case of risky debt, the promised yield is always greater than the risk-free rate since with a strictly positive SDF ($E_t[m_{t+1}\phi_{t+1}]R_t^f > 0$), the loss ratio is greater zero in at least one state but cannot become smaller than zero. In case ϕ_{t+1} is uncorrelated with the SDF, we can further simplify the expression for the yield rate to $r_t^c = r_t^f + E_t[\phi_{t+1}]$. Specifically, the yield rate is just the sum of the risk-free rate and the expected loss ratio. In case the loss ratio ϕ_{t+1} is correlated with consumption, we have to state an appropriate assumption with respect to their interrelation. A high loss ratio during times of low consumption and a low loss ratio in times when the investor's consumption is already high imply a negative correlation of losses with consumption. This leads to a positive correlation between the loss ratio and the SDF. Equation 2.3.5 shows that a stronger positive covariance of the loss ratio with the SDF increases the promised yield.

We define the return on debt as

$$R_{t+1}^D = \frac{Int_{t+1} + PP_{t+1}}{D_t} = \frac{RP_{t+1}}{D_t}. \quad (2.3.6)$$

By using the promised debt repayments $D_t(1 + r_t^c)$, the total losses on debt L_{t+1} and the loss ratio

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ϕ_{t+1} we are able to state

$$R_{t+1}^D = \frac{D_t(1 + r_t^c) - L_{t+1}}{D_t} = 1 + r_t^c - \phi_{t+1}. \quad (2.3.7)$$

For the case of risk-free debt, we observe that the absolute loss amounts to zero implying $r_t^c = r_{t+1}^D = r_t^f$. In case of risky debt, the net return on debt r_{t+1}^D is smaller than or equal to the promised yield. Taking expectations, we get the conditional expected return on debt, i.e., the discount rate for the expected sum of interest and principal payments

$$E_t[R_{t+1}^D] = 1 + r_t^c - E_t[\phi_{t+1}]. \quad (2.3.8)$$

For risky debt with $E_t[\phi_{t+1}] > 0$ we obtain $E_t[R_{t+1}^D] < 1 + r_t^c$, i.e., $E_t[r_{t+1}^D] < r_t^c$. Conditional expected returns on debt are equal to r_t^f for risk-free debt and for risky debt when the loss ratio is uncorrelated with the SDF: $E_t[R_{t+1}^D] = 1 + r_t^c - E_t[\phi_{t+1}] = 1 + r_t^f + E_t[\phi_{t+1}] - E_t[\phi_{t+1}] = 1 + r_t^f$.²⁹

Before we analyze the effect of the prioritization of debt payments on the tax shield value, it is useful to regard the interest and principal payments separately, and discuss their prices and returns as well as their relation to the price and return of the tax savings. We start by defining the return on interest payments R_{t+1}^{Int} , on principal payments R_{t+1}^{PP} and on the period specific tax savings R_{t+1}^{TS} :

$$R_{t+1}^{Int} = \frac{Int_{t+1}}{p_t(Int_{t+1})}, R_{t+1}^{PP} = \frac{PP_{t+1}}{p_t(PP_{t+1})}, R_{t+1}^{TS} = \frac{TS_{t+1}}{p_t(TS_{t+1})}. \quad (2.3.9)$$

Taking conditional expectations we get the respective conditional expected returns $E_t[R_{t+1}^{Int}]$, $E_t[R_{t+1}^{PP}]$ and $E_t[R_{t+1}^{TS}]$, i.e., the discount rates for conditional expected interest payments, principal payments and tax savings. Notice that those single payoffs are regularly not separately traded, even though there are some examples of such debt instruments, such as strip bonds and interest or principal-only classes of asset-backed securities.³⁰ With our set of assumptions we are able to state one important result of our analysis: The returns on tax savings are equal to the returns on interest payments $R_{t+1}^{Int} = R_{t+1}^{TS}$. The return on tax savings can be expressed in terms of the interest payments and their price, and by additionally noting that the tax rate cancels out we get

$$R_{t+1}^{TS} = \frac{TS_{t+1}}{p_t(TS_{t+1})} = \frac{\tau Int_{t+1}}{p_t(\tau Int_{t+1})} = \frac{\tau Int_{t+1}}{\tau p_t(Int_{t+1})} = R_{t+1}^{Int}. \quad (2.3.10)$$

²⁹For assets that are uncorrelated with the marginal utility growth of consumption of an investor (the SDF) the investor expects to earn a rate equal to the risk-free rate. Such an asset does not offer risk-reduction services and does not insure against bad states, i.e., low consumption, because on average it does not pay more in such a state. On average, it also does not pay more in good states, i.e., states of high consumption, so that the volatility of the consumption stream is not increased. See for an extensive treatment Magill and Quinzii (2002), p. 160.

³⁰If we would have additionally assumed the spanning assumption to hold, it is possible to perfectly duplicate the single payoffs of principal and interest payments on the capital market. See for a discussion of the spanning assumption e.g. Kruschwitz and Löffler (2006). However, we have abstained from making further assumptions with respect to the economy (see FN 7 and 11). As mentioned in Cochrane (2005) the basic pricing equation can also represent the private evaluation of non-traded assets.

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A proportional³¹ loss distribution on principal and interest payments means that losses on principal payments are given by $L_{t+1} \frac{D_t}{D_t(1+r_t^c)} = L_{t+1} \frac{1}{(1+r_t^c)}$ and losses on interest payments by $L_{t+1} \frac{r_t^c D_t}{D_t(1+r_t^c)} = L_{t+1} \frac{r_t^c}{(1+r_t^c)}$. In this case the returns of all debt cash flows are equal: $R_{t+1}^D = R_{t+1}^{Int} = R_{t+1}^{PP}$. The return on interest payments is given by $R_{t+1}^{Int} = \frac{Int_{t+1}}{p_t(Int_{t+1})} = \frac{r_t^c D_t - L_{t+1} \frac{r_t^c}{(1+r_t^c)}}{p_t(r_t^c D_t - L_{t+1} \frac{r_t^c}{(1+r_t^c)})}$. By multiplying the numerator and denominator by $\frac{1+r_t^c}{r_t^c}$, we obtain $R_{t+1}^{Int} = \frac{D_t(1+r_t^c) - L_{t+1}}{p_t(D_t(1+r_t^c) - L_{t+1})} = R_{t+1}^D$. Conducting a similar procedure for the return on principal payments reveals that $R_{t+1}^D = R_{t+1}^{PP}$.³² Notice that models in which in a default always the full promised amount $D_t(1+r_t^c)$ is lost have by construction a proportional loss distribution on interest and principal payments.³³ That is, for the prioritization assumptions to have any effect we need at least one default state in which a part of the promised debt repayments is redeemed.

With the prioritization assumptions (and partial default) we cannot expect that the equality of returns of the different debt cash flows still holds. In the following we analyze those cases.

2.3.3 Partial default and different priorities of payments to debtholders

Subsequently, we analyze the two mutually exclusive assumptions of interest or principal prioritization and their impact on the discount rate of the tax savings, i.e., the discount rate for interest payments.

2.3.3.1 Principal with higher priority than interest payments

In case of a prioritization of principal payments, losses on debt L_{t+1} are first absorbed by the promised interest payments $r_t^c D_t$. This implies that the actual interest payments are determined by

$$Int_{t+1} = \max(r_t^c D_t - L_{t+1}, 0). \quad (2.3.11)$$

We provide a graphical representation for the actual interest payments dependent on FCF_{t+1}^U in Figure 2.1. Therefore, we have to subsequently find a respective expression. Substituting equation 2.3.2 into 2.3.11 and using the definition of the FCF_{t+1}^L leads to $Int_{t+1} = \max(r_t^c D_t - \max(0, D_t(1+r_t^c) - FCF_{t+1}^U - \tau \cdot Int_{t+1}), 0)$. It follows, as long as the max-functions do not bind at zero we have $Int_{t+1} = D_t r_t^c - D_t(1+r_t^c) + FCF_{t+1}^U + \tau \cdot Int_{t+1}$, where the $r_t^c D_t$ terms cancel. Rearranging

³¹Proportional means pro rata according to promised interest and principal payments.

³²For returns on principal payments we write $R_{t+1}^{PP} = \frac{PP_{t+1}}{p_t(PP_{t+1})} = \frac{D_t - L_{t+1} \frac{1}{(1+r_t^c)}}{p_t(D_t - L_{t+1} \frac{1}{(1+r_t^c)})}$, multiply numerator and denominator by $1+r_t^c$ and arrive at $R_{t+1}^{PP} = \frac{D_t(1+r_t^c) - L_{t+1}}{p_t(D_t(1+r_t^c) - L_{t+1})} = R_{t+1}^D$.

³³For example Cooper and Nyborg (2008) conduct their analysis based on a full loss of principal and interest in default. However, without questioning their results, they use the promised yield as discount rate for the tax savings.

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for Int_{t+1} we obtain for the unbounded part of the function:

$$Int_{t+1} = \frac{1}{1-\tau}(FCF_{t+1}^U - D_t). \quad (2.3.12)$$

The interest payments have their lower bound at zero and their maximum at $r_t^c D_t$. This enables us to restate Int_{t+1} using max-functions dependent on FCF_{t+1}^U . We find the ‘‘strike’’ of these functions through first, setting $Int_{t+1} = 0$ and rearranging equation 2.3.12 for FCF_{t+1}^U , which yields $FCF_{t+1}^U = D_t$, and second, setting $Int_{t+1} = r_t^c D_t$ and rewriting the same equation to get to $FCF_{t+1}^U = D_t(1 + r_t^c(1 - \tau))$. This enables us to restate Int_{t+1} dependent on FCF_{t+1}^U through two max-functions with the respective strikes:

$$Int_{t+1} = \frac{1}{1-\tau}[\max(FCF_{t+1}^U - D_t, 0) - \max(FCF_{t+1}^U - D_t(1 + r_t^c(1 - \tau)), 0)]. \quad (2.3.13)$$

This payoff function is depicted in Figure 2.1 (solid line), where Int_{t+1} starts to increase as soon as $FCF_{t+1}^U > D_t$ and reaches its maximum at $r_t^c D_t$ for $FCF_{t+1}^U \geq D_t(1 + r_t^c(1 - \tau))$. Since paid interest incur tax savings τInt_{t+1} , the slope of the payoff of the interest is increased by a factor of $\frac{1}{1-\tau}$ in comparison to the FCF_{t+1}^U .

Principal payments can be similarly constructed by subtracting the losses on principal from D_t ³⁴

$$PP_{t+1} = D_t - \max(D_t - FCF_{t+1}^U, 0). \quad (2.3.14)$$

As shown in Figure 2.1, PP_{t+1} (dashed line) increases one-to-one with FCF_{t+1}^U . As soon as $FCF_{t+1}^U = D_t$ the promised principal payments are reached, i.e., the maximum value of the payoff diagram $PP_{t+1} = D_t$. Moreover, we observe from Figure 2.1 that the interest and principal payments are not proportional. As a consequence, their respective returns do not have the same distributions as in the the case of a proportional loss distribution.

We continue to analyze the expected returns of interest payments, which are at the same time the expected returns of tax savings, using the results of Coval and Shumway (2001), who found expected return relations between option expected returns and the expected returns of their underlying. Therefore, we restate equation 2.3.11 in terms of the total debt repayments by $Int_{t+1} = \max((D_t(1 + r_t^c) - L_{t+1}) - D_t, 0) = \max(RP_{t+1} - D_t, 0)$. This can be regarded as a call option payoff with strike D_t and underlying RP_{t+1} . In order to obtain an expression depending on expected returns, we divide equation 2.3.11 by the price of interest payments and take expectations. This procedure yields $E_t[RI_{t+1}] = \frac{E_t[\max(RP_{t+1} - D_t, 0)]}{p_t(\max(RP_{t+1} - D_t, 0))}$. By factoring out D_t and using $R_{t+1}^D = \frac{RP_{t+1}}{D_t}$, we obtain³⁵

$$E_t[RI_{t+1}] = \frac{E_t[\max(R_{t+1}^D - 1, 0)]}{p_t(\max(R_{t+1}^D - 1, 0))}. \quad (2.3.15)$$

³⁴Notice that principal payments only suffer losses when interest payments are already equal to zero.

³⁵We do not transform the gross return into net returns, i.e., $R - 1$ into r , because in the following we will regard the 1 as a strike of a call option. We continue to show an important consequence using this logic.

2.3. Risky debt with partial default - a single-period analysis

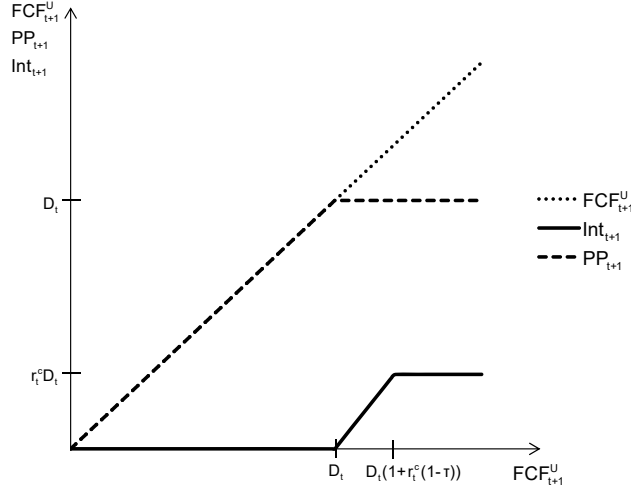


Figure 2.1: Payoffs of interest and principal payments dependent on the unlevered free cash flows for assumption A1.

Notice that with a strike equal to zero we would obtain $E_t[R_{t+1}^D]$ itself. Using the results of Coval and Shumway (2001), we can state qualitative relations of expected returns on debt and on interest payments. To apply their results for our case, we regard equation 2.3.15 with an arbitrary strike K :

$$E_t[R(K)_{t+1}] = \frac{E_t[\max(R_{t+1}^D - K, 0)]}{p_t(\max(R_{t+1}^D - K, 0))}. \quad (2.3.16)$$

Following Coval and Shumway (2001) and using conditional moments, the derivative with respect to K is given by³⁶

$$\begin{aligned} \frac{dE_t[R(K)_{t+1}]}{dK} &= \frac{-E_t[E_t[m_{t+1}|R_{t+1}^D](R_{t+1}^D - K)|R_{t+1}^D > K]}{(E_t[E_t[m_{t+1}|R_{t+1}^D](R_{t+1}^D - K)|R_{t+1}^D > K])^2} \\ &+ \frac{E_t[R_{t+1}^D - K|R_{t+1}^D > K]E_t[E_t[m_{t+1}|R_{t+1}^D]|R_{t+1}^D > K]}{(E_t[E_t[m_{t+1}|R_{t+1}^D](R_{t+1}^D - K)|R_{t+1}^D > K])^2}. \end{aligned} \quad (2.3.17)$$

The denominator is a square and therefore is always positive. The numerator is the negative of the conditional covariance $-Cov_t(R_{t+1}^D - K, E_t[m_{t+1}|R_{t+1}^D]|R_{t+1}^D > K)$. A negative (positive/zero) correlation yields a positive (negative/zero) numerator in equation 2.3.17, i.e., the expected return increases (decreases/remains the same) at that strike. With a strike equal to zero $E_t[R(K=0)_{t+1}]$ the expected return is $E_t[R_{t+1}^D]$. Increasing the strike K from zero to one constantly increases (decreases/keeps the same) the expected return when the conditional covariance is negative (positive/zero) for all values of K along this range. Furthermore, from equation 2.2.4 we know that a negative (positive/zero) correlation of R_{t+1}^D with the SDF implies an expected return greater than (smaller than/equal to) the risk-free rate. Since $E_t[R_{t+1}^D]$ can be regarded as a weighted average³⁷

³⁶For a full derivation see Coval and Shumway (2001).

³⁷ $E_t[R_{t+1}^D] = \frac{E_t[Int_{t+1}] + E_t[PP_{t+1}]}{D_t} = E_t[R_{t+1}^{Int}] \frac{p_t(Int_{t+1})}{D_t} + E_t[R_{t+1}^{PP}] \frac{p_t(PP_{t+1})}{D_t}$.

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of $E_t[R_{t+1}^{PP}]$ and $E_t[R_{t+1}^{Int}]$, when any of those expected returns is greater than $E_t[R_{t+1}^D]$ the other must be smaller.

From the stated relations we can derive the following proposition:³⁸

Proposition 1 Considering the case of a prioritization of principal payments and the aforementioned set of assumptions, in particular the tax deductibility of actually paid interest is always granted (main assumption 2), a possible COD will not be taxed by the tax authority (main assumption 3) the following relations between the expected return on tax savings and the expected return on debt always hold:

(i) For a negative conditional covariance $Cov_t(R_{t+1}^D - K, E_t[m_{t+1}|R_{t+1}^D]|R_{t+1}^D > K)$ for all $K \in [0, 1]$

$$E_t[R_{t+1}^{TS}] = E_t[R_{t+1}^{Int}] > E_t[R_{t+1}^D] > R_t^f, \quad (2.3.18)$$

(ii) for a positive conditional covariance $Cov_t(R_{t+1}^D - K, E_t[m_{t+1}|R_{t+1}^D]|R_{t+1}^D > K)$ for all $K \in [0, 1]$

$$E_t[R_{t+1}^{TS}] = E_t[R_{t+1}^{Int}] < E_t[R_{t+1}^D] < R_t^f, \quad (2.3.19)$$

(iii) for a zero conditional covariance $Cov_t(R_{t+1}^D - K, E_t[m_{t+1}|R_{t+1}^D]|R_{t+1}^D > K)$ for all $K \in [0, 1]$

$$E_t[R_{t+1}^{TS}] = E_t[R_{t+1}^{Int}] = E_t[R_{t+1}^D] = R_t^f. \quad (2.3.20)$$

Aggregate asset prices form the value of the market portfolio. That has to imply that the major part of the assets' individual values must move with the market, even though some might have a zero or a negative correlation with the aggregate market level. Coval and Shumway (2001) point out that existing asset pricing theory does not allow a positive correlation of the SDF with the market level. This indicates that the case of a negative correlation of the SDF with an individual asset's price is the dominant case (Proposition 1 (i)). Notice also that the propositions are "if" statements and not "if and only if" statements. For example the consequence of Proposition 1 (i) might still be true if the conditional covariance is not negative for all $K \in [0, 1]$ but only for parts of the interval. The propositions identify the cases where the results (the expected return relations) are always true.

³⁸Coval and Shumway (2001) assume no restrictions regarding the differentiability of the expected return with respect to K . We have discrete returns such that with increasing K at $K = R_{t+1}^D(\omega)$ the function is not smooth, i.e., it is not differentiable ($\max(R_{t+1}^D(\omega) - K, 0) = 0$ in this state). But the expected return is still continuous in K along the range of R_{t+1}^D , i.e., there are no jumps, and therefore the expected return will not suddenly decrease or increase. When we know that before and after such a point the derivative is positive, expected returns always increase with K .

2.3. Risky debt with partial default - a single-period analysis

2.3.3.2 Interest payments have priority over principal payments

The prioritization of interest payments implies that losses are first incurred on principal. By proceeding on this assumption, the interest payments dependent on the losses on debt are determined by

$$Int_{t+1} = r_t^c D_t - \max(L_{t+1} - D_t, 0), \quad (2.3.21)$$

where the max-function expresses the losses on interest payments. In comparison to the case of principal prioritization (e.g. equation 2.3.11), the actual interest payments start to decrease with increasing L_{t+1} if, and only if, the total losses L_{t+1} exceed the promised principal D_t . Rearranging equation 2.3.21 to a function depending on FCF_{t+1}^U (see Figure 2.2 for a graphical representation) yields

$$Int_{t+1} = r_t^c D_t - \max(r_t^c D_t - \frac{FCF_{t+1}^U}{1-\tau}, 0). \quad (2.3.22)$$

Observation of equation 2.3.22 reveals that losses on interest only emerge (at an amount of $\max(r_t^c D_t - \frac{FCF_{t+1}^U}{1-\tau}, 0)$) if $EBIT = \frac{FCF_{t+1}^U}{1-\tau}$ is smaller than the promised interest payments $r_t^c D_t$.

For completeness we state the formula for the principal payments. Here, considering the fact that prioritized interest generates tax savings, principal can be paid as soon as $FCF_{t+1}^U > r_t^c D_t(1-\tau)$ holds. Therefore, the payoff of the principal payments is given by

$$PP_{t+1} = \max(FCF_{t+1}^U - r_t^c D_t(1-\tau), 0) - \max(FCF_{t+1}^U - D_t(1+r_t^c(1-\tau)), 0). \quad (2.3.23)$$

Figure 2.2 illustrates that interest payments (solid line) increase with increasing $EBIT_{t+1} = \frac{FCF_{t+1}^U}{1-\tau}$. As a consequence, interest payments have a steeper slope than the unlevered free cash flows (dotted line). As soon as $EBIT_{t+1} = r_t^c D_t$ the actually incurred interest payments equal their promised value, i.e., $\frac{FCF_{t+1}^U}{1-\tau} = r_t^c D_t$. Afterwards, principal (dashed line) can be paid.

To make a qualitative statement regarding the expected returns on interest payments it is easier to get an expression for expected returns on principal payments first. From the alternative representation of principal payments, $PP_{t+1} = \max(RP_{t+1} - r_t^c D_t, 0)$, and taking out D_t we obtain the following expected return equation:

$$E_t[R_{t+1}^{PP}] = \frac{E_t[\max(R_{t+1}^D - r_t^c, 0)]}{p_t(\max(R_{t+1}^D - r_t^c, 0))}. \quad (2.3.24)$$

Again we follow Coval and Shumway (2001). Since $E_t[R_{t+1}^D]$ can be regarded as a weighted average of $E_t[R_{t+1}^{PP}]$ and $E_t[R_{t+1}^{Int}]$, when any of those expected returns is greater than $E_t[R_{t+1}^D]$ the other must be smaller. This implies the following proposition:

Proposition 2 Considering the case of a prioritization of interest payments and the foremen-

2.3. Risky debt with partial default - a single-period analysis

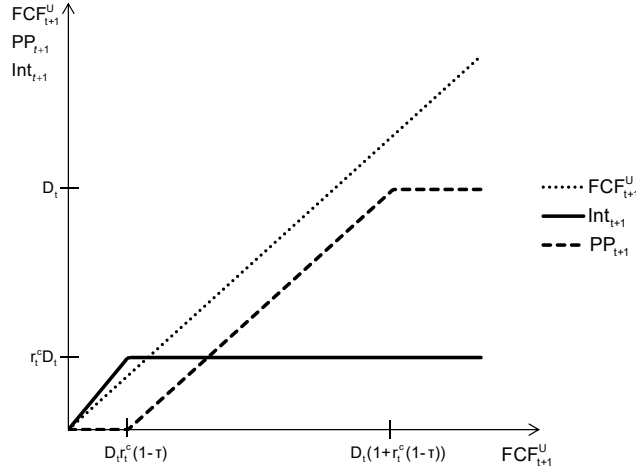


Figure 2.2: Payoffs of interest and principal payments relative to unlevered free cash flows for assumption A2.

tioned set of assumptions, in particular main assumption 2 and 3, the following relations between the expected return on tax savings and the expected return on debt always hold:

- (i) For a negative conditional covariance $Cov_t(R_{t+1}^D - K, E_t[m_{t+1}|R_{t+1}^D]|R_{t+1}^D > K)$ for all $K \in [0, r_t^c]$

$$E_t[R_{t+1}^{TS}] = E_t[R_{t+1}^{Int}] < E_t[R_{t+1}^D], \quad (2.3.25)$$

- (ii) for a positive conditional covariance $Cov_t(R_{t+1}^D - K, E_t[m_{t+1}|R_{t+1}^D]|R_{t+1}^D > K)$ for all $K \in [0, r_t^c]$

$$E_t[R_{t+1}^{TS}] = E_t[R_{t+1}^{Int}] > E_t[R_{t+1}^D], \quad (2.3.26)$$

- (iii) for a zero conditional covariance $Cov_t(R_{t+1}^D - K, E_t[m_{t+1}|R_{t+1}^D]|R_{t+1}^D > K)$ for all $K \in [0, r_t^c]$

$$E_t[R_{t+1}^{TS}] = E_t[R_{t+1}^{Int}] = E_t[R_{t+1}^D] = R_t^f. \quad (2.3.27)$$

In contrast to Proposition 1 (i), Proposition 2 (i) shows for the case of a prioritization of interest payments that the expected return on interest and in turn on the tax savings is smaller than the one of the overall debt issue. For this case interest payments carry less risk than the overall debt issue. Interest are paid first, i.e. losses are applied to interest last.

Overall, comparing our findings summarized in Propositions 1 and 2 with the case of risk-free debt or risky debt combined with a taxation of a COD (see e.g. Sick (1990) or Kruschwitz et al. (2005)) the prioritization assumptions have an important impact on the appropriate discount rate (expected return) of the tax savings. Moreover, our findings indicate given that the debtor aims at maximizing its firm value, a prioritization of interest payments has a favorable discount rate effect on the value of tax savings. To get a sense for the consequences on discount rates, promised yield rates and the value of the tax shield of applying either one or the other assumption on the same

firm in terms of its assets we provide an illustrative example.

2.3.3.3 Illustrative example

We construct a numerical example with a correlation between the debt repayments and the SDF, so that the results of Proposition 1 (i) and 2 (i) hold. We model $N = 100$ states with uniform probabilities $\pi(\omega) = 1/100$ for $\omega = 1, 2, \dots, 100$. The state-dependent values for $EBIT_{t+1}$ range from 5 to 30, with the in-between values being evenly spaced. The corporate tax rate amounts to $\tau = 25\%$ and the gross risk-free rate to $R_t^f = 1.05$. We vary the debt value D_t stepwise (50 steps) from 0 to $D_t = V_t^L$, i.e., a leverage of one. The correlation between EBIT and the SDF³⁹ is -0.812 .

For the following exemplary analysis, we will first show how to determine several necessary values which is followed by a step-by-step analysis of the discount rates of tax savings, the promised yield and the value of the tax shield. In order to be able to compare the results, we use for the presented figures as ratio for the abscissa D_t/V_t^U . Notice that V_t^U remains constant with increasing debt levels, whereas V_t^L does not. The levered firm value V_t^L implied by the two assumptions is not equivalent for the same total amount of debt and in turn would result in different D_t/V_t^L for the same debt level. Using D_t/V_t^U additionally enables us to observe how much the firm can be levered above the unlevered firm value under the different assumptions.

Using the pricing equation $V_t^U = E_t[m_{t+1}FCF_{t+1}^U]$ we determine for the unlevered firm a value rounded to three digits of $V_t^U = 11.312$ and $E_t[R^U] = 1.16$. We find this value by multiplying the state-dependent SDF (as discussed in FN 39), $m_{t+1}(\omega)$, with the respective unlevered free cash flows, $FCF_{t+1}^U(\omega)$, summing up all values and dividing by the number of states, here 100, to account for probabilities. Moreover, we can determine the maximum debt level $D_t^{rf\ max}$ that can still be regarded as (default-) risk-free, i.e., in every state the firm is able to fully repay interest and redemption. By using equation 2.3.2, setting $L_{t+1}|_{D_t=D_t^{rf\ max}} = 0$, noting that in this case $r_t^c = r_t^f$ has to hold and rearranging for $D_t^{rf\ max}$, the maximum risk-free debt level is determined by $D_t^{rf\ max} = \frac{EBIT(\omega=1)(1-\tau)}{1+(1-\tau)r_t^f}$, where $EBIT(\omega = 1) = 5$ are the lowest possible earnings in this example. The maximum risk-free debt level is given by $D_t^{rf\ max} = 3.615$. For values of $D_t/V_t^U \leq 3.615/11.312 = 31.953\%$ debt is risk-free. Therefore, we observe in the subsequently discussed figures that debt starts to carry default-risk for $D_t/V_t^U > 31.953\%$.

³⁹The EBIT increases linearly from 5 to 30 from state 1 to state 100. We assume that the investor's consumption increases linearly from state 1 to 100, so that we can replace the ω indicating states with consumption C_{t+1} , i.e., $C_{t+1}(\omega = 1) = 1, C_{t+1}(\omega = 2) = 2, \dots, C_{t+1}(\omega = 100) = 100$. We assume an SDF of the form $m_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}$, where $\beta = \frac{1}{R_t^f}$ and $u'(\cdot)$ is the first derivative of a utility function $u(\cdot)$ (see

Cochrane (2005) for a basic treatment). We use the power utility $u(C_{t+1}) = \frac{C_{t+1}^{1-\gamma}}{1-\gamma}$, where the first derivative is $u'(C_{t+1}) = C_{t+1}^{-\gamma}$. Thus, the SDF is $m_{t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$. The constant γ is the coefficient of risk-aversion and must be positive for risk-averse investors. We set $\gamma = 0.25$ and $C_t = 33.686$. With numeric values the SDF is $m_{t+1} = \frac{1}{1.05} \left(\frac{C_{t+1}}{33.686}\right)^{-0.25}$. The SDF is a convex function over states and consumption with values from $m_{t+1}(\omega = 1) = 2.294$ to $m_{t+1}(\omega = 100) = 0.726$.

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Irrespective of the assumed priority assumption, for determining the discount rate for the tax savings according to $E_t[R_{t+1}^{TS}] = E_t[R_{t+1}^{Int}] = \frac{E_t[Int_{t+1}]}{p_t(Int_{t+1})}$ we have to obtain values for $E_t[Int_{t+1}]$ and $p_t(Int_{t+1})$. $E_t[Int_{t+1}]$ depends on the promised yield, r_t^c , and the loss, L_{t+1} . Since L_{t+1} depends on r_t^c and Int_{t+1} while r_t^c itself depends on L_{t+1} we observe a feedback, where one cannot solve for one variable due to the max-functions. Therefore, we solve numerically. The results of the analysis are presented and discussed subsequently.

Figure 2.3 depicts the discount rate for the tax savings and debt issue implied by either assuming a principal ($E_t[R_{t+1}^{Int}](A1)$, solid line / $E_t[R_{t+1}^D](A1)$, long dashes) or interest prioritization ($E_t[R_{t+1}^{Int}](A2)$, dotted line / $E_t[R_{t+1}^D](A1)$, short dashes) dependent on D_t/V_t^U . By assuming a prioritization of principal payments we observe at a value of approximately 31.95% for D_t/V_t^U that the discount rate for the tax savings starts to increase above the risk-free rate r_t^f . Moreover, we observe a very steep increase of the discount rate for the tax savings with increasing values for D_t/V_t^U . In states with a comparably low EBIT, where the promised debt repayments cannot be paid anymore, losses are first incurred on interest payments. With increasing leverage, in an increasing number of states, interest cannot or not fully be paid. The negative covariance of returns on interest payments with the SDF decreases even more implying an increase of the discount rate of the tax savings. Assuming a prioritization of interest payments (Figure 2.3, dotted line) reveals a completely different picture. We observe that interest payments and thereby the tax savings only become risky for very high values of D_t/V_t^U . Obviously, this is a direct result of the fact that losses are first assigned to principal payments. Only if losses exceed the principal of D_t , tax savings due to interest payments are affected.

The example also underlines the qualitative results made in the prior two subsections. The negative correlation between the SDF and the EBIT leads here also to a negative correlation between the debt repayments and the SDF. First, as soon as debt becomes risky, the discount rates of debt for the different prioritization rules start to differ. This is a direct consequence of the fact that the incurred state-dependent tax savings differ as well. Second, we observe for principal prioritization a discount rate of the tax savings strictly exceeding the one for the overall debt issue (long dashes). In contrast to this, for a prioritization of interest the discount rate of the tax savings is strictly lower than the corresponding one for debt (small dashes).

Additionally, we analyze the promised yield implied by the prioritization rule (see Figure 2.4). On the one hand, the promised yield affects the interest payments and, in turn, the upper bound of the tax savings. On the other hand, it is indirectly affected by the actually incurred interest and tax savings since both influence the losses on the debt repayments. Figure 2.4 depicts the promised yields implied by principal (curve A1, solid line) or interest prioritization (curve A2, dotted line). Both yield rates exceed the risk-free rate at the same value for D_t/V_t^U and increase with increasing D_t/V_t^U . We observe a constantly lower promised yield for the case of interest prioritization than for principal prioritization. In contrast to the latter case, with interest prioritization the firm is still able to generate tax savings at higher values of D_t/V_t^U , which in turn can be used to pay principal.

2.3. Risky debt with partial default - a single-period analysis

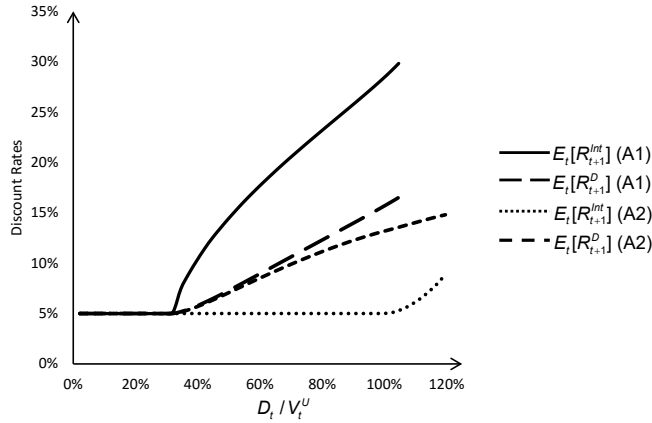


Figure 2.3: Discount rates of tax savings depending on D_t/V_t^U .

This leads to comparably lower losses and to a lower promised yield. Moreover, similar to Figure 2.3, we observe that for the case of interest prioritization (A2) the firm is able to maintain debt financing for values $D_t/V_t^U > 1$ since higher tax shield values are generated. This leads us to the next step of our analysis.

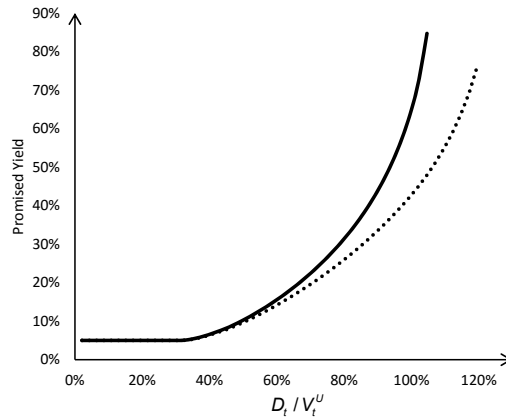


Figure 2.4: Promised Yield for principal and interest prioritization dependent on D_t/V_t^U .

Figure 2.5 shows the effect of the prioritization rules on the tax shield value. Obviously, with increasing values for D_t/V_t^U both tax shield values increase. Approximately, at a value for D_t/V_t^U of 31.95%, i.e., when debt becomes risky, the implied tax shield values start to differ. By assuming interest prioritization (curve A2, dotted line) the value is strictly higher than in the case of principal prioritization (curve A1, solid line). Even though the promised yield for the case of interest prioritization is comparably lower at equivalent values of D_t/V_t^U (see Figure 2.4), interest are paid in more states and therefore, tax savings are generated in more states. The effect of the relatively lower discount rate for interest prioritization is much stronger on the tax shield value than the one of lower promised interest payments. The relatively small tax shield value for the case of principal prioritization is a direct result of the high interest payments in states when a unit of cash flow has a low value and low interest payments in states when a unit of cash flow has a high

2.4. Risky debt and partial default on interest payments - a multi-period analysis

value. Higher discount rates and a lower value compared to the case with interest prioritization are the consequence (compare Figure 2.3 and 2.5). Finally, due to the higher tax shield value under the assumption of interest prioritization, it is possible to lever the firm far more over its unlevered firm value.

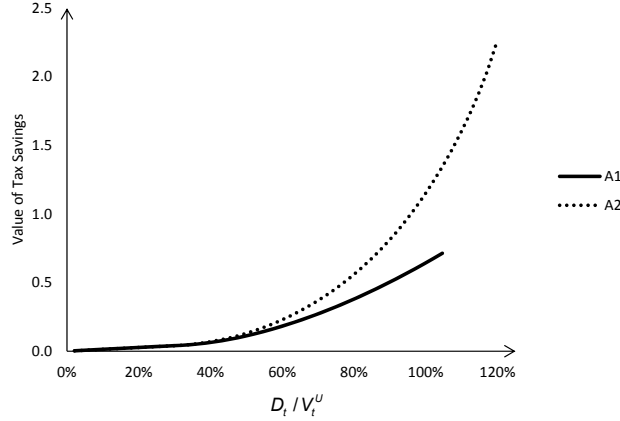


Figure 2.5: Value of the tax shield for principal and interest prioritization dependent on D_t/V_t^U .

2.4 Risky debt and partial default on interest payments - a multi-period analysis

2.4.1 Additional assumptions and definitions for the multi-period case

We extend our time frame from two points in time, t and $t + 1$, towards T , with $s = t, t + 1, \dots, T$ points in time. Therefore, we have to adjust and extend our current set of assumptions with some remaining valid for the extended time frame, e.g. the main assumptions 1 to 3. The considered firm exists until period T . The firm pursues a constant leverage financing policy, which implies that the firm issues or redeems a corresponding debt amount (net principal payments PP_s^{net}) at an arbitrary time point s in order to keep leverage constant. We regard only single-period debt contracts. The promised yield is determined by the debtholders at the beginning of each period.

In a multi-period setting, we have to state an adequate assumption for the capital structure imposed after the occurrence of a default. Typically, we could differentiate between three possibilities: Either the firm is liquidated for its salvage value, sold to a new investor and/or restructured to be able to carry debt again. With respect to this, we assume that after a default, independent of the new owner, the same capital structure is established as ex ante to default. This implies that a constant leverage is reestablished.⁴⁰

By extending our model framework towards a multi-period setting we have to discuss the reason of default. In the single-period analysis the firm defaulted when it was unable to pay the contractually fixed interest and principal payments. In a multi-period setting we have to

⁴⁰See for example Leland (1994), Goldstein et al. (2001) or Koziol (2014), who discuss this in more detail. Kruschwitz and Löffler (2005), p. 228, and Rapp (2006), p. 778, build their analysis on the equivalent assumption.

additionally consider the value of the firm which could be used to satisfy the debtholders. Several bankruptcy codes distinguish between two types of default: illiquidity (cash flow insolvency) and overextension (balance sheet insolvency). The bankruptcy code of the USA in section 101 (32)(A) defines the term “insolvency” as default, where the entity’s debt exceeds its property measured at fair value. Nevertheless, the United States Court of Appeals for the Ninth Circuit recently noted in the matter of Marshall III, No. 09-55573, 2013 WL 3242478 (9th Cir. June 28, 2013) that if insolvency was a prerequisite to a chapter 11 filing, the cash flow test would be an appropriate default criterion as well (see section 101 (32)(C)).⁴¹ Due to this differentiation, we define both default triggers based upon our set of assumptions, discuss their implications and choose the most appropriate one for our analysis.

Based upon our assumptions that the tax authority always grants the deductibility of the actually paid interest payments and that after default the firm pursues again the ex ante default financing policy, we can conclude that the firm is supposed to be illiquid according to the following definition.⁴²

Definition of illiquidity: The levered firm is illiquid at an arbitrary time $s + 1$ if the levered cash flows plus debt issued at time $s + 1$ are not sufficient to cover the sum of promised interest and principal payments of the prior debt issue:

$$FCF_{s+1}^L + D_{s+1} < (1 + r_s^c)D_s. \quad (2.4.1)$$

In the presented model illiquidity will be resolved by refinancing as long as the firm is not overextended. Basically, illiquidity could be financed by a sale of assets or issuance of equity. In case of an additional equity issue the raised amount is directly paid to the debtholders. The liquidation of assets decreases the unlevered free cash flows by selling some of the productive assets. Obviously, the last consequence is not compatible with our assumption that the unlevered free cash flows are independent of the pursued financing policy of the firm.⁴³ Here, we proceed on the assumption that illiquidity is always resolved by refinancing. Having noted this we concentrate on the default trigger for overextension.

Definition of overextension: The levered firm defaults due to overextension if the promised debt payments $D_s(1 + r_s^c)$ are greater than the current levered free cash flows FCF_{s+1}^L and the value of all future levered free cash flows V_{s+1}^L :

$$FCF_{s+1}^L + V_{s+1}^L < (1 + r_s^c)D_s. \quad (2.4.2)$$

⁴¹Other bankruptcy codes as for example in Germany distinguish between overextension and illiquidity as well. In the UK the Supreme Court clarified in a decision published 9th of May 2013 UKSC 28 that the cash flow and balance sheet test are both applicable for testing insolvency.

⁴²See for an equivalent definition of illiquidity Kruschwitz et al. (2005). Additionally, note that this condition can be rearranged to $ECF_t < 0$. This implies that the firm defaults as soon as the cash flows distributed to equity holders turn negative, i.e., the equityholders have to inject cash into the firm in order to prevent a default.

⁴³See for a discussion on this issue Kim et al. (1993).

2.4. Risky debt and partial default on interest payments - a multi-period analysis

The default trigger has to be consistent with the assumption of the financing policy after default. Since we assumed that the leverage will be kept after default, the firm defaults as soon as the levered value of the firm and the respective cash flow is exceeded by the promised debt repayments.

The subsequent two sections show how the multi-period setting affects the payoffs of the interest and principal payments dependent on the prioritization assumptions. Since we use single-period debt, we can keep all the results of 2.3.2 just using s and $s + 1$ as time subscripts. Furthermore, we adjust the payoff equations of the single-period analysis conducted in 2.3.3.

2.4.2 Principal with higher priority than interest payments

The equations for interest 2.4.3 and principal payments 2.4.4 differ from the single-period equations only with respect to the future levered firm value V_{s+1}^L , which is added to the unlevered free cash flows. The extended equation for interest payments is then

$$\begin{aligned} Int_{s+1} = & \frac{1}{1-\tau} \left[\max(FCF_{s+1}^U + V_{s+1}^L - D_s, 0) \right. \\ & \left. - \max(FCF_{s+1}^U + V_{s+1}^L - D_s(1 + r_s^c(1-\tau)), 0) \right], \end{aligned} \quad (2.4.3)$$

and for principal payments

$$PP_{s+1} = D_s - \max(D_s - (V_{s+1}^L + FCF_{s+1}^U), 0). \quad (2.4.4)$$

In comparison to the single-period setting the levered firm value V_{s+1}^L can be interpreted as an increase of FCF_{s+1}^U , i.e., V_{s+1}^L can cover losses as well, whereas V_{s+1}^L is a mix of debt $D_{s+1} = lV_{s+1}^L$ and equity $S_{s+1} = (1-l)V_{s+1}^L$. Figure 2.6 shows the payoffs for interest (Int_{s+1}^* , dash-dotted line) and principal payments (PP_{s+1}^* , long dashed) for the multi-period and single-period (Int_{s+1} , solid line / PP_{s+1} , short-dashed line) analysis in order to highlight the differences. The levered firm value V_{s+1}^L causes both payoffs to shift to the left by the dashed area. For ease of presentation and to show the principle, we assign the same levered firm value to each unlevered free cash flow. Note that with different state-dependent unlevered free cash flows the implied levered firm values should differ respectively.⁴⁴

2.4.3 Interest with higher priority than principal payments

The payoff functions only change through the additional V_{s+1}^L . The equation for interest payments⁴⁵ is

$$Int_{s+1} = r_s^c D_s - \max\left(r_s^c D_s - \frac{FCF_{s+1}^U + V_{s+1}^L}{1-\tau}, 0\right), \quad (2.4.5)$$

⁴⁴Notice that putting $FCF_{s+1}^U + V_{s+1}^L$ on the axes leads to the same graph as in the single-period case.

⁴⁵In equation 2.4.5 V_{s+1}^L is also divided by $(1-\tau)$ because within the range of zero and $r_s^c D_s$ interest payments are $Int_{s+1} = FCF_{s+1}^L + V_{s+1}^L = FCF_{s+1}^U + \tau Int_{s+1} + V_{s+1}^L$. Rearranging this for interest payments leads to $Int_{s+1} = \frac{1}{1-\tau}(FCF_{s+1}^U + V_{s+1}^L)$.

2.4. Risky debt and partial default on interest payments - a multi-period analysis

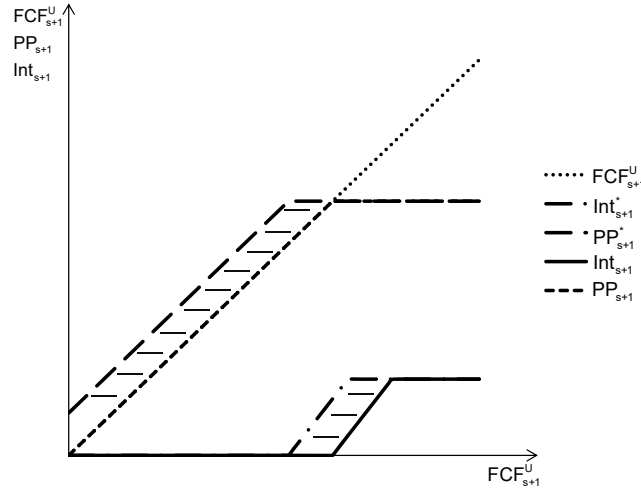


Figure 2.6: Interest and principal payments with principal priority (single-period and multi-period case).

and for principal payments

$$PP_{s+1} = \max(FCF_{s+1}^U + V_{s+1}^L - D_s r_s^c(1 - \tau), 0) - \max(FCF_{s+1}^U + V_{s+1}^L - D_s(1 + r_s^c(1 - \tau)), 0). \quad (2.4.6)$$

Figure 2.7 illustrates for interest prioritization the payoff functions for the multi-period setting for interest (Int_{s+1}^* , dash-dotted line) and principal payments (PP_{s+1}^* , long dashed) depending on FCF_{s+1}^U . Again, in order to compare the multi-period and single-period payoffs we depict the latter as well (Int_{s+1} , solid line / PP_{s+1} , short-dashed line) and are able to note that the payoff functions shift to the left by the amount of V_{s+1}^L (dashed area), which is held constant across states. With respect to the shift, the explanation is equivalent to the case of principal prioritization.

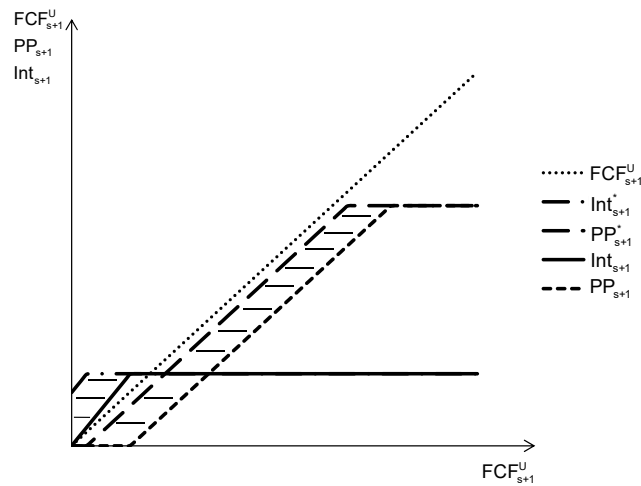


Figure 2.7: Interest and principal payments with interest priority (single-period and multi-period case).

2.4. Risky debt and partial default on interest payments - a multi-period analysis

2.4.4 Priority assumptions and the Miles and Ezzell result

Miles and Ezzell (1985) (ME) regard the valuation of the tax savings for the case of constant leverage and risk-free debt. They found that the value of tax savings at time t is obtained by discounting the expected tax savings over one period at the risk-free rate and for the remaining periods until t at the same discount rate as for the unlevered free cash flows. Subsequently, we will show that an equivalent result can be obtained for risky debt and the discussed priority assumptions for interest and principal payments.

For the herein discussed analysis we keep the assumptions already made in this section. Additionally, we assume *independently and identically distributed (i.i.d.) returns on unlevered free cash flows* R^U and an *i.i.d. SDF*. We drop time subscripts for i.i.d. random variables. We define R^U as single period returns by

$$R^U = \frac{p_{i+1}(FCF_{s+1}^U)}{p_i(FCF_{s+1}^U)}, \quad (2.4.7)$$

for $s = t, t+1, \dots, T-1$, $i = t, t+1, \dots, s$. In Appendix 2.6 we establish that the i.i.d. assumptions for the SDF and the returns of the unlevered free cash flows are in line with the assumptions of the multi-period model of Miles and Ezzell (1985).

As the first step, we specify the *objective of the subsequent analysis*. The value of the tax shield can be determined as the sum of prices of the period specific tax savings $VTS_t = \sum_{s=t}^{T-1} p_t(TS_{s+1})$. For the period from the time $s+1$ when a tax saving TS_{s+1} is incurred to time s we can use the results obtained in our single-period analysis. For this period the tax savings are proportional to interest payments and are therefore discounted by $E_s[R_{s+1}^{Int}]$, i.e., $p_s(TS_{s+1}) = \frac{E_s[TS_{s+1}]}{E_s[R_{s+1}^{Int}]}$. For the ME-result to remain valid, we have to additionally show that $p_t(p_s(TS_{s+1})) = \frac{E_t[p_s(TS_{s+1})]}{E_t[R^U]^{s-t}}$ for $s = t, t+1, \dots, T-1$, or restated

$$\frac{E_t[p_s(TS_{s+1})]}{p_t(p_s(TS_{s+1}))} = E_t[R^U]^{s-t}. \quad (2.4.8)$$

Furthermore, for equation 2.4.8 to hold, it is sufficient to demonstrate that

$$\frac{p_s(TS_{s+1})}{p_t(p_s(TS_{s+1}))} = (R^U)^{s-t} \quad (2.4.9)$$

holds⁴⁶. If $p_s(TS_{s+1})$ is proportional to a variable that has to be discounted by R^U over any period (denote this value $p_s(CF_{s+1}^{R^U})$), then $p_s(TS_{s+1})$ also has to be discounted by R^U over any period.⁴⁷

⁴⁶Notice that with i.i.d. R^U we can write $E_t[(R^U)^{s-t}] = E_t[R^U]^{s-t}$

⁴⁷To see that we write $p_s(TS_{s+1}) = cp_s(CF_{s+1}^{R^U})$, where c is a constant, and take prices $p_t(\cdot)$ for $t < s$ to obtain

$p_t(p_s(TS_{s+1})) = cp_t(CF_{s+1}^{R^U})$, which is $p_t(TS_{s+1}) = cp_t(CF_{s+1}^{R^U})$. Since $\frac{cp_s(CF_{s+1}^{R^U})}{cp_t(CF_{s+1}^{R^U})} = (R^U)^{s-t}$, where the constants cancel out, $\frac{p_s(TS_{s+1})}{p_t(TS_{s+1})} = (R^U)^{s-t}$ must hold as well.

Thus, it is sufficient to show that

$$\frac{p_s(TS_{s+1})}{p_s(CF_{s+1}^R)} = \text{constant} \quad (2.4.10)$$

holds for $s = t, \dots, T - 1$.

As the second step, we include the *constant leverage assumption* and *introduce an adjusted leverage*. The leverage l is given by

$$l = \frac{D_s}{V_s^L} = \frac{D_s}{V_s^U + VTS_s} = \frac{D_s}{V_s^U + p_s(VTS_{s+1}) + p_s(TS_{s+1})}. \quad (2.4.11)$$

Additionally, we define an adjusted leverage ratio, for any $s = t, t + 1, \dots, T - 1$, by

$$l_s^* = \frac{D_s}{V_s^U + p_s(VTS_{s+1})}. \quad (2.4.12)$$

This ratio excludes the price of TS_{s+1} from the denominator of leverage. We work with ratios because they are independent from level values such as FCF^U and V^U , which can have different distributions over any single period, so that we would have to consider many different cases. Instead, we work with R^U , which has only a single distribution over any single period due to the i.i.d. assumption, so that we only have to consider one case. The adjusted leverage will come handy for converting level values such as V_s^U and FCF_s^U into ratios⁴⁸. We use this conversion in the period specific analyses below.

As the third step, we analyze *period* $T - 1$ to T for both prioritization assumptions. For this period, the unlevered firm value is given by $V_{T-1}^U = p_{T-1}(FCF_T^U)$ and $p_{T-1}(VTS_T) = 0$. Therefore, the adjusted leverage is given by $l_{T-1}^* = \frac{D_{T-1}}{p_{T-1}(FCF_T^U)}$. By dividing the numerator and denominator of the leverage ratio $l = \frac{D_{T-1}}{p_{T-1}(FCF_T^U) + p_{T-1}(TS_T)}$ by $p_{T-1}(FCF_T^U)$ we obtain

$$l = \frac{l_{T-1}^*}{1 + \frac{p_{T-1}(TS_T)}{p_{T-1}(FCF_T^U)}}. \quad (2.4.13)$$

Since $p_{T-1}(FCF_T^U)$ has to be discounted by R^U over any single period, according to relation 2.4.10 we have to show that $p_{T-1}(TS_T)$ is proportional to $p_{T-1}(FCF_T^U)$, i.e., $\frac{p_{T-1}(TS_T)}{p_{T-1}(FCF_T^U)} = \text{constant}$. Since l is a constant per assumption, with $\frac{p_{T-1}(TS_T)}{p_{T-1}(FCF_T^U)}$ being a constant, l_{T-1}^* must be a constant as well. Or conversely, given constant leverage l and given constant l_{T-1}^* the ratio $\frac{p_{T-1}(TS_T)}{p_{T-1}(FCF_T^U)}$ must be a constant. We have to show that the other direction works as well, i.e., with given i.i.d. R^U , i.i.d. m , and constant l , l_{T-1}^* is a constant, i.e., is unique. With several l_{T-1}^* possible for a single leverage the ratio $\frac{p_{T-1}(TS_T)}{p_{T-1}(FCF_T^U)}$ can have several values as well. We will continue to show that l and l_{T-1}^* have a one-to-one relation, i.e., for one leverage there is a single adjusted leverage from which follows that $\frac{p_{T-1}(TS_T)}{p_{T-1}(FCF_T^U)} = \text{constant}$.

⁴⁸Notice that returns are also ratios.

2.4. Risky debt and partial default on interest payments - a multi-period analysis

Subsequently, we analyze the *prioritization of principal payments for the period $T - 1$ to T* .

Using $l_{T-1}^* = \frac{D_{T-1}}{p_{T-1}(FCF_T^U)}$ we rewrite the equation for the ratio $\frac{p_{T-1}(TS_T)}{p_{T-1}(FCF_T^U)}$ as

$$\frac{p_{T-1}(TS_T)}{p_{T-1}(FCF_T^U)} = \tau l_{T-1}^* \frac{p_{T-1}(Int_T)}{D_{T-1}} = \tau l_{T-1}^* \frac{D_{T-1} - p_{T-1}(PP_T)}{D_{T-1}} \quad (2.4.14)$$

and use equation 2.4.4 and $\frac{FCF_T^U}{D_{T-1}} = \frac{p_{T-1}(FCF_T^U)FCF_T^U}{p_{T-1}(FCF_T^U)D_{T-1}} = \frac{R^U}{l_{T-1}^*}$ to write

$$\begin{aligned} \frac{p_{T-1}(PP_T)}{D_{T-1}} &= p_{T-1} \left(1 - \max \left(1 - \frac{R^U}{l_{T-1}^*}, 0 \right) \right) \\ &= p_{T-1} \left(\frac{R^U}{l_{T-1}^*} - \max \left(\frac{R^U}{l_{T-1}^*} - 1, 0 \right) \right). \end{aligned} \quad (2.4.15)$$

We rewrite equation 2.4.13 to

$$l = \frac{l_{T-1}^*}{1 + \tau l_{T-1}^* \left(1 - p_{T-1} \left(\frac{R^U}{l_{T-1}^*} - \max \left(\frac{R^U}{l_{T-1}^*} - 1, 0 \right) \right) \right)}, \quad (2.4.16)$$

and divide by l_{T-1}^* and simplify to obtain

$$l = \frac{1}{\frac{1}{l_{T-1}^*} (1 - \tau) + \tau + \tau p_{T-1} \left(\max \left(\frac{R^U}{l_{T-1}^*} - 1, 0 \right) \right)}. \quad (2.4.17)$$

We observe that there are only positive constants connected to the $\frac{1}{l_{T-1}^*}$ terms. We conclude that for $l_{T-1}^* \in (0, l_{T-1, max}^*]$, where $l_{T-1, max}^*$ is the adjusted leverage at which leverage is 100%, leverage strictly monotonously increases with adjusted leverage, i.e., the two variables have a one-to-one relation. For one leverage there is one adjusted leverage and vice versa, and the ratio $\frac{p_{T-1}(TS_T)}{p_{T-1}(FCF_T^U)}$ is a constant given leverage l . That means $p_{T-1}(TS_T)$ has to be discounted by R^U over any period from $T - 1$ back to t .

Next, we analyze the *prioritization of interest payments for the period $T - 1$ to T* . We use $l_{T-1}^* = \frac{D_{T-1}}{p_{T-1}(FCF_T^U)}$ and equation 2.4.5 to write the ratio $\frac{p_{T-1}(TS_T)}{p_{T-1}(FCF_T^U)}$ as

$$l_{T-1}^* \frac{p_{T-1}(TS_T)}{D_{T-1}} = l_{T-1}^* \tau p_{T-1} \left(r_{T-1}^c - \max \left(r_{T-1}^c - \frac{R^U}{l_{T-1}^* (1 - \tau)}, 0 \right) \right). \quad (2.4.18)$$

After taking similar steps as in the prior case, i.e., rearranging equation (2.4.13) using (2.4.18), the leverage for interest prioritization is given by

$$l = \frac{1}{\frac{1}{l_{T-1}^*} + \frac{p_{T-1}(TS_T)}{D_{T-1}}} = \frac{1}{\frac{1}{l_{T-1}^*} + \tau p_{T-1} \left(r_{T-1}^c - \max \left(r_{T-1}^c - \frac{R^U}{l_{T-1}^* (1 - \tau)}, 0 \right) \right)}. \quad (2.4.19)$$

One cannot immediately observe the relation between adjusted leverage and leverage since this equation includes the promised yield rate as well, which also depends on l_{T-1}^* . This implies, that we additionally have to find out how a change of adjusted leverage changes the promised yield.

2.4. Risky debt and partial default on interest payments - a multi-period analysis

From equation 2.3.5 we state the equation for the promised yield for the respective period:

$$r_{T-1}^c = r^f + R^f E_{T-1}[m\phi_T]. \quad (2.4.20)$$

We have to look for a representation of r_{T-1}^c in terms of l_{T-1}^* . By using equation 2.3.2 for the losses and dividing them by D_{T-1} we obtain the loss ratio:

$$\phi_T = \max\left(1 + r_{T-1}^c - \frac{FCF_T^U}{D_{T-1}} - \frac{TS_T}{D_{T-1}}, 0\right). \quad (2.4.21)$$

Using again $\frac{FCF_T^U}{D_{T-1}} = \frac{R^U}{l_{T-1}^*}$ we obtain

$$\phi_T = \max\left(1 + r_{T-1}^c - \frac{R^U}{l_{T-1}^*} - \frac{TS_T}{D_{T-1}}, 0\right). \quad (2.4.22)$$

We substitute in the equation for the ratio $\frac{TS_T}{D_{T-1}}$ that we know from equation 2.4.18

$$\begin{aligned} \phi_T = \max\left(1 + r_{T-1}^c - \frac{R^U}{l_{T-1}^*} - \tau\left(r_{T-1}^c \right. \right. \\ \left. \left. - \max\left(r_{T-1}^c - \frac{R^U}{l_{T-1}^*(1-\tau)}, 0\right)\right), 0\right), \end{aligned} \quad (2.4.23)$$

and simplify to

$$\begin{aligned} \phi_T = \max\left(1 + r_{T-1}^c(1-\tau) - \frac{R^U}{l_{T-1}^*} \right. \\ \left. + \tau \max\left(r_{T-1}^c - \frac{R^U}{l_{T-1}^*(1-\tau)}, 0\right), 0\right). \end{aligned} \quad (2.4.24)$$

The loss ratio is greater than zero, i.e., does not bind, in default states only. The more default states there are the higher is the impact of a change of l_{T-1}^* on r_{T-1}^c . We regard the limiting case when leverage is one and a change of l_{T-1}^* changes the loss ratio in every state and therefore has the biggest impact on r_{T-1}^c . That means we can leave out the outer max-function in 2.4.24 because the term inside this max-function is never smaller than zero. There is no state in which a cash flow goes to equityholders. The inner max-function describes the losses of interest payments divided by D_{T-1} . Interest will not be in default in every state at a leverage of one since losses are first applied to principal payments⁴⁹. Thus, we leave out the outer max-function and use equation 2.4.20 to obtain

$$r_{T-1}^c = R^f + r_{T-1}^c(1-\tau) - \frac{R^f}{l_{T-1}^*} + \tau R^f p_{T-1} \left(\max\left(r_{T-1}^c - \frac{R^U}{l_{T-1}^*(1-\tau)}, 0\right) \right). \quad (2.4.25)$$

⁴⁹The converse is always true: When interest payments experience losses, the loss ratio must not bind, i.e., must not be zero.

We rearrange and simplify to

$$\tau r_{T-1}^c = R^f \left(1 - \frac{1}{l_{T-1}^*} \right) + \tau R^f p_{T-1} \left(\max \left(r_{T-1}^c - \frac{R^U}{l_{T-1}^*(1-\tau)}, 0 \right) \right). \quad (2.4.26)$$

By substituting equation 2.4.26 into 2.4.19 we observe that the prices of the max-functions cancel out and we obtain:

$$l = \frac{1}{\frac{1}{l_{T-1}^*} + \frac{p_{T-1}(TS_T)}{D_{T-1}}} = \frac{1}{\frac{1}{l_{T-1}^*} + \left(1 - \frac{1}{l_{T-1}^*} \right)} = 1. \quad (2.4.27)$$

At the highest possible leverage of one a change in $\frac{1}{l_{T-1}^*}$ is exactly offset by the change of the term $\frac{p_{T-1}(TS_T)}{D_{T-1}}$ from equation 2.4.19. With respect to equation 2.4.13 that means that a change of l_{T-1}^* is exactly offset by the change in $1 + \frac{p_{T-1}(TS_T)}{p_{T-1}(FCF_T^U)}$ at $l = 1$. At lower leverages and lower adjusted leverages there will be at least one state without losses, i.e., states where the outer max-function of the loss ratio binds so that the loss ratio is zero. Thus, with a lower leverage than one, changes of $\frac{1}{l_{T-1}^*}$ will not be fully offset by changes of the term $\frac{p_{T-1}(TS_T)}{D_{T-1}}$. Any decrease of $\frac{1}{l_{T-1}^*}$, i.e., an increase in l_{T-1}^* , leads to a smaller increase of the term $\frac{p_{T-1}(TS_T)}{D_{T-1}}$, so that the denominator of l decreases and l increases. Thus, leverage strictly monotonously increases with adjusted leverage for $l_{T-1}^* \in (0, l_{T-1, max}^*]$. Again, the ratio $\frac{p_{T-1}(TS_T)}{p_{T-1}(FCF_T^U)}$ is a constant given leverage l , and $p_{T-1}(TS_T)$ has to be discounted by R^U over any period from $T - 1$ back to t .

As the fourth and last step, we conduct the *analysis for the periods $T - 2$ to $T - 1$ and prior periods*. Again, we have to consider both prioritization assumptions. We state the leverage ratio as

$$l = \frac{l_{T-2}^*}{1 + \frac{p_{T-2}(TS_{T-1})}{V_{T-2}^U + p_{T-2}(p_{T-1}(TS_T))}}. \quad (2.4.28)$$

Since we have learned that $p_{T-1}(TS_T)$ has to be discounted by R^U over any single period from $T - 1$ to t , we observe that the expression in the denominator of the ratio $\frac{p_{T-2}(TS_{T-1})}{V_{T-2}^U + p_{T-2}(p_{T-1}(TS_T))}$ has to be discounted by R^U . The objective is to show that this ratio is a constant, which again implies that l_{T-2}^* is a constant or conversely, with constant l_{T-2}^* the ratio is a constant, for given l .

Again, we start with the case of a *prioritization of principal payments*. Using $l_{T-2}^* = \frac{D_{T-2}}{V_{T-2}^U + p_{T-2}(p_{T-1}(TS_T))} = \frac{D_{T-2}}{V_{T-2}^U + p_{T-2}(V_{T-1}^L)}$ we rewrite the equation for the ratio $\frac{p_{T-2}(TS_{T-1})}{V_{T-2}^U + p_{T-2}(V_{T-1}^L)}$ as

$$\begin{aligned} \frac{p_{T-2}(TS_{T-1})}{V_{T-2}^U + p_{T-2}(V_{T-1}^L)} &= \frac{\tau l_{T-2}^* p_{T-2}(Int_{T-1})}{D_{T-2}} \\ &= \tau l_{T-2}^* \frac{D_{T-2} - p_{T-2}(PP_{T-1})}{D_{T-2}}. \end{aligned} \quad (2.4.29)$$

We again use equation 2.4.4 and $\frac{FCF_{T-1}^U + V_{T-1}^L}{D_{T-2}} = \frac{FCF_{T-1}^U + p_{T-1}(FCF_T^U) + p_{T-1}(TS_T)}{D_{T-2}} =$

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$\frac{R^U (V_{T-2}^U + p_{T-2}(p_{T-1}(TS_T)))}{D_{T-2}} = \frac{R^U}{l_{T-1}^*}$ to write

$$\begin{aligned} \frac{p_{T-2}(PP_{T-1})}{D_{T-2}} &= p_{T-2} \left(1 - \max \left(1 - \frac{R^U}{l_{T-2}^*}, 0 \right) \right) \\ &= p_{T-2} \left(\frac{R^U}{l_{T-2}^*} - \max \left(\frac{R^U}{l_{T-2}^*} - 1, 0 \right) \right). \end{aligned} \quad (2.4.30)$$

We obtain exactly the same equation as for the last period except for a change of subscripts. That means the analysis from period $T - 1$ to T can be applied similarly. The equation for leverage for the respective period is then

$$l = \frac{1}{\frac{1}{l_{T-2}^*}(1 - \tau) + \tau + \tau p_{T-2} \left(\max \left(\frac{R^U}{l_{T-2}^*} - 1, 0 \right) \right)}. \quad (2.4.31)$$

Since there is a one-to-one relation between leverage and adjusted leverage plus constants, distributions are the same as in the last period which implies that adjusted leverage must have the same value as in the last period for the same leverage. Therefore, the ratio $\frac{p_{T-2}(TS_{T-1})}{V_{T-2}^U + p_{T-2}(VTS_{T-1})}$ must also be the same as in the last period. Similarly, one can go back until the period t to $t + 1$.

We conclude that constant leverage, i.i.d. R^U and i.i.d. SDF imply a constant adjusted leverage over all states and also over time as well as a constant ratio $\frac{p_s(TS_{s+1})}{V_s^U + p_s(VTS_{s+1})}$ over all states and over time.

Let us finally regard the case of a *prioritization of interest payments* for the periods $T - 2$ to $T - 1$ and prior. We use equation 2.4.5 and $l_{T-2}^* = \frac{D_{T-2}}{V_{T-2}^U + p_{T-2}(VTS_{T-1})}$ to write the ratio $\frac{p_{T-2}(TS_{T-1})}{V_{T-2}^U + p_{T-2}(VTS_{T-1})}$ as

$$l_{T-2}^* \frac{p_{T-2}(TS_{T-1})}{D_{T-2}} = l_{T-2}^* \tau p_{T-2} \left(r_{T-2}^c - \max \left(r_{T-2}^c - \frac{R^U}{l_{T-2}^*(1 - \tau)}, 0 \right) \right). \quad (2.4.32)$$

Again, we face the same expression as in the period $T - 1$ to T ; only the subscripts changed. Substituting the aforementioned equation into the equation for leverage also yields the same equation as in the last period. The conclusions are the same: The ratio $\frac{p_{T-2}(TS_{T-1})}{V_{T-2}^U + p_{T-2}(VTS_{T-1})}$ as well as adjusted leverage are constant across states for a given leverage. Furthermore, following the same reasoning as for the prior case adjusted leverage is constant over time and across states and so is the ratio $\frac{p_s(TS_{s+1})}{V_s^U + p_s(VTS_{s+1})}$.

In the last step we make some *final remarks*. With constant leverage, i.i.d. SDF and i.i.d. R^U adjusted leverage is constant and prices of tax savings have always to be discounted at R^U . We can write any debt related return (R^D, R^{PP}, R^{Int}) as functions of i.i.d. SDF, i.i.d. R^U and constants. Any such function must also be i.i.d. With i.i.d. R^{Int} , we can write $E[R^{Int}]$ without time subscripts as the constant expected return of interest payments over any period. Finally, we

2.5. Conclusion

can state a valuation equation similar to equation 19 of Miles and Ezzell (1985):

$$V_t^L = \sum_{i=1}^{T-t} \frac{E_t[FCF_{t+i}^U]}{E[R^U]^i} + \sum_{i=1}^{T-t} \frac{E_t[TS_{t+i}]}{E[R^U]^{i-1}E[R^{Int}]}. \quad (2.4.33)$$

Even though there are nonlinearities introduced through the prioritization of either interest or principal payments, assuming an i.i.d. SDF and i.i.d. unlevered returns still allow us to use a valuation equation that shows a structure similar to the one of Miles and Ezzell (1985).

2.5 Conclusion

We aimed at analyzing how tax savings have to be discounted under the assumption of a partial default on debt. We identified the risk structure of the tax savings by finding factors influencing its major driver the tax-deductible interest payments. Therefore, we have focussed on corporate taxation and relied on a set of assumptions that aims at approximating the current tax settings in legislations such as the US or Germany. By separating the payments to debtholders into interest and principal, we first analyzed the impact of default on the tax savings in a single-period setting. We generally found under the presented set of assumptions the tax savings to have a risk structure equivalent to the one of the interest payments, and therefore both should have the same discount rate (i.e., conditional expected return). As a major factor driving the risk of the interest payments and in turn the tax savings we identified the prioritization of either principal or interest in case of a partial default. In contrast to Molnár and Nyborg (2013), who differentiated between the prioritization rules as well, simple assumptions regarding the SDF enabled us to derive qualitative statements for the discount rate of the tax savings, (interest,) principal payments, the overall debt issue and their interrelation.

With a negative conditional covariance of debt repayments with the SDF over certain ranges of debt repayments, a prioritization of principal payments implied a higher discount rate of the tax savings than the one of the principal payments and the overall debt issue. For a prioritization of interest payments the contrary is true: The tax savings showed a comparably lower risk and in turn a lower discount rate. Only in the very narrow case of a proportional loss distribution the discount rates of the tax savings, the debt issue, the interest and principal payments were always equivalent, independent of any assumption about correlations with the SDF.

By using the results of the single-period analysis, we have extended our analysis towards a multi-period setting. Assuming i.i.d. returns, i.i.d. SDF and a constant leverage we derived a tax shield pricing equation considering the risk of default which shows a structure equivalent to the one obtained by Miles and Ezzell (1985). There is just one difference. Instead of discounting the tax savings one period by the risk-free rate one has to discount one period by the expected return on interest payments.

Summarizing, equivalent to works like Rapp (2006) or Cooper and Nyborg (2008) we find default to have a non-neglectable impact on the discount rate of the tax savings and in turn on the

2.6. Appendix - Cash flow and return processes in the multi-period model

overall tax shield value. Extending the aforementioned works we demonstrate how to include both prioritization possibilities. In contrast to other works such as Molnár and Nyborg (2013) we do not rely on a two-state model where the recovery rate in default has to be known with certainty. More generally, our result depends on the distributions and correlations with the SDF.

These findings enable us to draw several implications for topics related to the pricing of tax savings. Our findings imply that simply using the “cost of debt” for discounting the tax savings when faced with a prioritization rule might bias the tax shield value significantly. Though, when the use of the “cost of debt” for valuation purposes is desired we have shown the proportional loss distribution to be in most of the cases the necessary precondition. Without a proportional loss distribution the appropriate discount rate for the tax savings is equivalent to the one for the overall debt issue only in special cases. This implies that either the legal practice concerned with defining the bond conditions accounts for this respectively or the business practice of valuing tax savings has to draw more attention to using the appropriate discount rate for the tax savings.

Nevertheless, we have to explicitly draw upon the limitations of the presented framework. For example, as already outlined above, we do not include personal taxes in our analysis. Therefore, we do not regard the consequences of a default on the level of debtholders and equityholders. On the side of the debtholder a debt write-down might be tax-deductible to a certain extent. Since this lowers the debtholders loss in the event of a default, comprising this tax consequence into the debtholder’s calculus might have an impact on the promised yield and in turn on the discount rate of the tax savings. Differentiating between commercial banks and private debtholders is desirable. In this respect, additional research should be conducted. Furthermore, since the tax savings are a relevant factor for analyzing the optimal capital structure of the firm, it might be an interesting issue to analyze the impact of interest or principal prioritization.

2.6 Appendix - Cash flow and return processes in the multi-period model

To obtain single-period discount rates for unlevered equity that are constant over time and constant among different cash flows Miles and Ezzell (1985) assume an expectation revision process (see also Fama (1977)) for the unlevered free cash flows given by

$$E_{u+1}[FCF_{s+1}^U] = E_u[FCF_{s+1}^U](1 + \epsilon_{u+1}), \quad (2.6.1)$$

where $E_u[\epsilon_{u+1}] = 0$ ⁵⁰, for $s = t, t + 1, \dots, T - 1$ and $t \leq u \leq s$. For $u = s$, i.e., the period s to $s + 1$, we obtain $FCF_{s+1}^U = E_s[FCF_{s+1}^U](1 + \epsilon_{s+1})$. The single-period expected return for $u = s$ can be determined by dividing the right-hand side of equation 2.6.1 and its expectation, respectively, by

⁵⁰Take conditional expectations $E_u[\cdot]$ of both sides of equation 2.6.1.

its price⁵¹:

$$R_{s+1}^U = \frac{1 + \epsilon_{s+1}}{p_s(1 + \epsilon_{s+1})}, E_s[R_{s+1}^U] = \frac{1}{p_s(1 + \epsilon_{s+1})}. \quad (2.6.2)$$

Miles and Ezzell (1985) assume a constant covariance between ϵ and the SDF⁵² over time. Together with a constant risk-free rate and constant conditional expectations of ϵ the price $p_s(\epsilon_{s+1})$ is constant over time⁵³. This implies for $E_s[R_{s+1}^U]$ to be constant over time and across different cash flows.

Miles and Ezzell (1985) make no explicit assumptions about the distributions of returns. Since they price with a multi-period CAPM Fama (1977) states that returns have to be independent and investment opportunities have to be constant over time to not invalidate the pricing process of the CAPM.⁵⁴ Therefore we assume that ϵ and the SDF are independently and identically distributed (i.i.d.).⁵⁵ R^U is also i.i.d. since it is a function of i.i.d. ϵ and i.i.d. SDF. We drop the time subscripts of these variables in the following. I.i.d. processes have constant conditional expectations so that in the case of i.i.d. returns single-period conditional expected values can be used as single-period discount rates.⁵⁶ Discounting a cash flow FCF_s^U over two periods leads to

$$\frac{FCF_{s+1}^U}{p_{s-1}(FCF_{s+1}^U)} = \frac{(1 + \epsilon)(1 + \epsilon)}{p((1 + \epsilon)(1 + \epsilon))} = \frac{(1 + \epsilon)}{p(1 + \epsilon)} \frac{(1 + \epsilon)}{p(1 + \epsilon)} = R^U R^U, \quad (2.6.3)$$

whereas one can similarly go further back in time. We can write the second equality due to the i.i.d. assumption. Taking expected values leads to constant single-period discount rates. An i.i.d. SDF implies constant single-period risk-free rates.⁵⁷

⁵¹The $E_s[FCF_{s+1}^U]$ terms cancel out.

⁵²Miles and Ezzell (1985) base their analysis on the CAPM. In case the covariance of the market return and ϵ is constant over time, the SDF is a linear combination of the market return.

⁵³Using equation 2.2.1 we are able to state $p_s(\epsilon_{s+1}) = E_s[m_{s+1}\epsilon_{s+1}] = E_s[m_{s+1}]E_s[\epsilon_{s+1}] + cov_s(m_{s+1}, \epsilon_{s+1})$ and observe that with constant $E_s[m_{s+1}]$, $E_s[\epsilon_{s+1}]$ and $cov_s(m_{s+1}, \epsilon_{s+1})$ the price of ϵ is constant over time.

⁵⁴They refer to the traditional Sharpe-Lintner-Black CAPM.

⁵⁵A sequence of a random variable X_s , for $s = t, t + 1, \dots, T$, is i.i.d. if any term in the sequence has the same cumulative probability distribution function, i.e. $F_{X_s}(x) = F_{X_v}(x)$ for all realisations x and all $s, v \in [t, t + 1, \dots, T]$, and if the joint cumulative distribution function of the sequence equals the product of all single cumulative distribution functions, i.e. $F_X(x_t, x_{t+1}, \dots, x_T) = F_{X_t}(x_t)F_{X_{t+1}}(x_{t+1}) \dots F_{X_T}(x_T)$.

⁵⁶For a further discussion of discount rates in a multi-period CAPM setting see Fama (1977).

⁵⁷Notice that $R^f = E[m]$.

Chapter 3

Valuation effects of taxes on debt cancellation¹

¹This chapter is based on Krause and Lahmann (2017).

Bringing this publication into context

This article follows chronologically and logically the first article on the discount rates of tax savings. The main focus is now on an additional assumption. When debt is cancelled in a default, the cancelled amount may be subject to taxes. That is, the cancelled amount may be seen as corporate income and therefore is taxed at the tax rate of corporate profits. Throughout the work on this article, we found that the corporate finance literature usually assumes that cancelled debt is always taxed. In some cases, this assumption is not stated but implicitly used. We give a detailed discussion on those issues. However, for example in the U.S., both cases are possible, so that presupposing a taxation of cancelled debt gives a wrong picture. With an eye on practical issues of valuation, we derive tax-adjusted discount rates and WACC-like discount rates for both cases, a taxation of cancelled debt and no taxation of cancelled debt. In the second case, the usual textbook equations can only generally be used for a proportional (pro-rata) distribution of losses on interest and principal payments.

As mentioned, this paper follows naturally from the first one, in that it extends some of the issues of tax shield valuation. The third paper of this dissertation, which is on the de- and re-levering procedure of betas, finds a new field of application for the conclusions made herein.

Abstract

Standard models on firm valuation regard a simplified default setting, often not revealing relevant implicit assumptions. In this paper, we analyze the impact of risky debt and of taxes on a cancellation of indebtedness (COD) on tax savings. For the case of a taxation of a COD, we explicitly show that the risky components in the pricing equation of tax savings cancel out so that the tax shield pricing is similar to the case of risk-free tax savings. Furthermore, assuming no tax on a COD, we show the standard textbook equations for the tax shield, the Tax-adjusted discount rates and WACC subject to risky debt to be generally valid only for a pro-rata loss distribution between interest and principal payments. Using standard equations for the case of no taxes on a COD in case of a non-proportional loss distribution can lead to substantial misvaluations, which we illustrate with an example.

Keywords: Default Risk, Tax Treatment of Default, Tax Savings, Tax-adjusted Discount Rates

3.1 Introduction

The appropriate pricing of interest tax savings is a relevant element in firm valuation. Many articles show (e.g., Graham, 2000; Kemsley and Nissim, 2002) the interest tax shield to be a significant factor of the enterprise value. Therefore, identifying the value drivers of the interest tax shield is generally important.

In the light of the long history of the discounted cash flow (DCF) literature discussing the pricing of levered firms (e.g., using tax-adjusted discount rates or the weighted average cost of capital) or separately of interest tax shields, it might be surprising that the impact of default, or in other terms, risky debt, is still under discussion. When faced with risky debt, the pricing of the tax savings differs substantially from the procedure introduced by the classic works, such as Modigliani and Miller (1958) / (1963) and Miles and Ezzell (1980) / (1985), with respect to the assumptions, the tax treatment of a default, and its structural modelling. More recently, some authors discuss one of the most important value drivers in tax shield pricing, the tax treatment of a cancellation of indebtedness (COD) (see e.g., Kruschwitz and Löffler, 2006; Cooper and Nyborg, 2008; Blaufus and Hundsdorfer, 2008).

A COD might give rise to a certain tax liability: The tax authority might regard the written-down amount of debt, i.e., the cancelled amount of debt or COD, as an extraordinary income which is subject to corporate taxes. Whether a COD is taxed or not depends on the tax code and whether the firm seeks for reorganization or not. Each case has a specific implication on the pricing of tax savings.

Sick (1990) already considers the impact of risky debt on the pricing of levered firms assuming a taxation of a possible COD. He shows the tax-adjusted discount rates (TADR) for risky debt to have an equivalent structure as for the case of risk-free debt.² Cooper and Nyborg (2008) revive the discussion with a specific focus on the pricing of the interest tax savings for risky debt and the tax treatment of a COD. In contrast to Sick (1990), they focus on the derivation of TADRs assuming that a possible COD is not taxed. Particularly, they regard a two-state binomial model where the firm completely defaults in one state (i.e., the debtholders incur a full loss on their promised interest and principal payments) and remains solvent in the other. Both results are important for practical valuation settings because the correct tax treatment of a COD depends on the tax jurisdiction. Quite clearly, this may be one of the reasons why we observe a clear cut in the literature stream in articles favoring the taxation of a COD (e.g., Sick, 1990; Kruschwitz and Löffler, 2006) and those favoring the contrary assumption (e.g., Cooper and Nyborg, 2008; Molnár and Nyborg, 2013; Krause and Lahmann, 2015). Several articles address the impact of default on the tax savings and their pricing (e.g., Couch et al., 2012; Koziol, 2014) and implicitly assume a certain tax treatment of a COD. Still, there are some works proposing pricing equations which are not consistent to the possible cases of a tax treatment of default. However, Arzac and Glosten

²Kruschwitz and Löffler (2006) obtain an equivalent result.

(2005) touch this important aspect but propose a pricing equation that is not consistent with all of their mentioned special cases for risky debt because the COD tax treatment is not clear.

The aim of our paper is threefold. First, we consider a setting with risky debt where we model both cases of the COD tax treatment. We explicitly show that the risky elements in the tax shield pricing equation cancel out for the case that taxes on a COD are applicable. Second, for the case of no taxes on a COD, we emphasize the often disregarded fact that the pricing of interest tax savings directly depends on the loss distribution among interest and principal payments. Thereby, we reveal the well-known and established pricing equations for the tax savings to be generally valid only for a proportional or pro-rata loss distribution. Third, we derive a WACC pricing equation that is consistent to the often regarded case of no taxes on a COD and that substantially differs from the standard textbook formulas. Moreover, we carefully test under which conditions the expected return on debt (often referred to as cost of debt) proposed by several authors (e.g., Modigliani and Miller, 1963; Myers, 1974; Miles and Ezzell, 1980) could be used as a parameter within tax-adjusted discount rates and the WACC equation. We find that this is only possible for risk-free debt or for debt without systematic risk, i.e., only cases in which the expected return on debt equals the risk-free rate.

The remainder of this article is structured as follows. In Section 3.2 we briefly outline the model basics. Section 3.3 analyzes the effects of the tax treatment of a COD and a proportional loss distribution on the pricing of the interest tax shields. The effect on the structure of tax-adjusted discount rates is derived in Section 3.4. Besides this, we aim at uncovering the effect on the textbook WACC. In Section 3.5, we numerically show the bias occurring when the standard equations implicitly assuming a pro-rata loss distribution are applied for the case of an interest prioritization. Section 3.6 concludes our analysis.

3.2 Model basics

We regard a single-period model with only two points in time, t and $t + 1$. Limiting the time horizon enables us to focus on the effects of a COD. In a multi-period setting we would have to discuss other issues such as the optimal bankruptcy level, the possible continuance of the firm after default, or the recapitalization / liquidation strategy following a default. Since we regard only t and $t + 1$, we maintain a simple notation by omitting time subscripts.

A firm generates an uncertain and unlevered cash flow stream at time $t+1$. The state-dependent free cash flow values are given by $FCF(\omega)$, where ω denotes the state with $\omega \in [0, N]$ and $0 < FCF(0) < FCF(1) < \dots < FCF(N)$. We assume an arbitrage-free capital market and the existence of a price for the subsequently discussed cash flows, i.e., all cash flows are tradable. By stating this assumption, we abstain from discussing the pricing of non-tradeable cash flows with different valuations among market participants. This allows us to use a single pricing operator, $p(\cdot)$, denoting the price at time t of a future, $t + 1$, cash flow. For example, $p(FCF)$ denotes the price of the unlevered free cash flow and thus, constitutes the unlevered firm value at t . Applying

3.2. Model basics

the pricing operator on a risk-free, i.e., deterministic, cash flow of one yields the inverse of the gross risk-free rate, i.e., $p(1) = \frac{1}{R_f}$. We either state returns as gross R or net values r , where $R = 1 + r$. $E[\cdot]$ denotes the expected value operator contingent on the available information at time t .

At time t the firm issues debt at par with a value of D and is obliged to pay at $t + 1$ interest payments amounting to $r^c D$, where r^c denotes the promised yield rate contractually fixed at time t , and D the promised principal payments. In case of risk-free debt, the debtholders' contractually fixed promised yield is equivalent to the risk-free rate, i.e., $r^c = r_f$. At this point of our analysis, we omit the case of risky debt and refer to Section 3.3.

The corporate tax rate, τ , is assumed to be constant and deterministic and the corporate tax allows for an immediate full loss offset. We only consider corporate taxes and abstain from modelling personal taxes (as for example, Modigliani and Miller, 1958; Miles and Ezzell, 1980; Cooper and Nyborg, 2008; Krause and Lahmann, 2015). Introducing personal taxes does not contribute to our analysis, which aims at uncovering the effect-relationship of the tax treatment of a COD. The simplified tax setting avoids mixing up our result with other issues as for example the tax treatment of a COD on the level of the debtholders. Finally, regardless of a possible default, we assume that the tax authority grants the tax deductibility of the actually paid interest at time $t + 1$.³

The value of an otherwise identical but levered firm, V^L , whose assets generate at time $t + 1$ an unlevered cash flow, $FCF(\omega)$, can be represented as the sum of the equity and debt value

$$V^L = S + D, \quad (3.2.1)$$

where S denotes the value of equity. Alternatively, we may determine the levered firm value according to the adjusted present value (APV) approach (see e.g., Modigliani and Miller, 1963; Myers, 1974), by

$$V^L = V^U + VTS, \quad (3.2.2)$$

where V^U denotes the unlevered firm value and VTS the value of the interest tax shield. Note that we retain the classic Modigliani and Miller assumption that the $EBIT$ is independent of the firm's financing decisions and of a possible default.

In line with several representatives of the tax shield literature (e.g., Kruschwitz and Löffler, 2006; Arzac and Glosten, 2005; Krause and Lahmann, 2015), we define the tax savings as difference between the taxes of an unlevered firm, Tax^U , and the taxes of a levered firm, Tax^L . Formally, in

³This assumption has been explicitly stated by Kruschwitz and Löffler (2005) and implicitly by Molnár and Nyborg (2013).

3.3. Tax shield, tax treatment of a COD and proportional loss distribution

our single period framework, the tax savings at time $t + 1$ are determined by

$$TS = Tax^U - Tax^L. \quad (3.2.3)$$

First, note that the unlevered firm pays taxes amounting to $Tax^U = \tau EBIT$, where $EBIT$ denotes the earnings before interest and taxes. With the aforementioned assumptions, FCF and $EBIT$ relate by $FCF = EBIT(1 - \tau) - Inv$, where Inv are the net investments. Now, we focus on the taxes paid by a levered firm, Tax^L . So far, without having discussed the consequences of a default, we can only show the result for risk-free debt. In this case, the interest payments reduce the $EBIT$ such that $Tax^L = \tau(EBIT - r_f D)$ holds. Thus, the tax savings generated at time $t + 1$ amount to

$$TS = \tau r_f D. \quad (3.2.4)$$

By using the price operator $p(\cdot)$ and noting the fact that Equation (3.2.4) only comprises certain quantities, we determine the price of the tax savings, i.e., the tax shield (value), as $p(TS) = p(\tau r_f D) = \frac{\tau r_f D}{1+r_f}$. Without stating any further assumptions for the case of risky debt, we can only acknowledge that the tax savings depend on the interest payments in case of default and the tax treatment of a possible COD.

3.3 Tax shield, tax treatment of a COD and proportional loss distribution

In the following sections, we discuss the impact of risky debt on the tax savings. More precisely, we focus on the tax treatment of a COD and explicitly show the relation to the distribution of losses among interest and principal payments that debtholders may face. For the often implicitly regarded case (e.g., Cooper and Nyborg, 2008) of a proportional loss distribution, we show how to calculate the tax shield values.

3.3.1 The impact of the tax treatment of a COD

Subsequently, we regard the case of a default and its impact on tax savings. In our single-period analysis, the firm defaults as soon as the available cash flows are insufficient to cover the promised interest and principal payments at time $t + 1$. The firm either partially or completely defaults on its debt obligations. In case of a default, the actual interest payments amount to

$$Int = r^c D - L^{Int}, \quad (3.3.1)$$

where L^{Int} denotes the losses on interest payments with $L^{Int} \in [0, r^c D]$, and the actual principal payments to

$$PP = D - C, \quad (3.3.2)$$

3.3. Tax shield, tax treatment of a COD and proportional loss distribution

where $C \in [0, D]$ denotes the losses on principal payments, i.e., the COD. Adding both yields the total losses on debt $L = L^{Int} + C$.

The cancelled amount of principal payments might induce a certain tax consequence. Basically, the tax authority might regard the cancelled amount, C , as taxable income. For example, in the United States (US) a COD gives rise to a tax liability according to IRC section 61(a)(12).⁴ But the US tax code also grants an exemption. In the case of liabilities exceeding assets, i.e., the typical balance sheet insolvency, the firm is able to avoid the tax liability by reorganizing through chapter 11. Other tax jurisdictions such as the United Kingdom have similar rules (see CTA (2009) section 354). In Germany, the income tax code explicitly states that even in the case of an insolvency, the cancellation of indebtedness gives rise to a tax liability (§11 KStG Nr. 7). Nevertheless, according to a letter of the German Ministry of Finance on the 27th of March 2003 and the BFH decision as of the 14th of June 2010, X R 34/08, an exemption from taxation of a COD might be granted if the firm aims for restructuring. Summarizing, depending on whether the firm aims at a restructuring process or not, a COD is taxable or not.⁵ Therefore, we analyze the impact of both cases on the tax shield.

Following the aforementioned discussion, in case taxes are not applicable on a COD, the levered firm pays at time $t + 1$ taxes equal to $Tax^L = \tau EBIT - \tau Int$. Whenever taxes on a COD apply, the tax payments amount to $Tax^L = \tau EBIT - \tau Int + \tau C$, where τC represents the taxes on a COD. By substituting these tax payments in Equation (3.2.3), we obtain the tax savings for the case of no taxes on a COD (NC) and for the case of taxes on a COD (C):

$$TS^{NC} = \tau Int, \quad (3.3.3)$$

$$TS^C = \tau Int - \tau C. \quad (3.3.4)$$

Table 3.1 provides an overview of the state-dependent interest and principal payments as well as the tax savings depending on the tax treatment of a COD.

Our pricing technique allows us to determine the value of the tax savings at time $t + 1$ by

$$VTS^{(\cdot)} = p(TS^{(\cdot)}), \quad (3.3.5)$$

where the superscript (\cdot) is a placeholder for the two cases: No taxation of a COD (NC) and taxation of a COD (C). By applying the price operator on the Equations (3.3.3) and (3.3.4), we

⁴See Miller (1991) for an extensive discussion of the tax consequences of corporate restructuring. Cooper and Nyborg (2008) discuss this issue in context of deriving tax-adjusted discount rates.

⁵Notice that both cases have been already separately discussed in the literature. For example, Kruschwitz and Löffler (2006) proceed on the assumption of a taxation of a possible COD according to §11 KStG Nr. 7. In contrast to this, Cooper and Nyborg (2008), Molnár and Nyborg (2013) and Krause and Lahmann (2015) regard the case of no taxes on a COD. In particular, Cooper and Nyborg (2008) and Molnár and Nyborg (2013) use as underlying justification the exemption by the US tax code (IRC section 61(a)(12)). According to Cooper and Nyborg (2008), the result obtained by Sick (1990) implicitly proceeds on the assumption of a taxation of a possible COD and therefore is not consistent with the results of Miles and Ezzell (1980).

3.3. Tax shield, tax treatment of a COD and proportional loss distribution

| Payments: | State-dependent payoffs: | |
|-------------|--------------------------|--|
| | No default | Default |
| Interest | $r^c D$ | $r^c D - L^{Int}$ |
| Principal | D | $D - C$ |
| Total | $(1 + r^c)D$ | $(1 + r^c)D - L^{Int} - C$ |
| Tax savings | $\tau r^c D$ | $\tau(r^c D - L^{Int})$ $\tau(r^c D - L^{Int}) - \tau C$ |

Table 3.1: State-dependent Debt and Tax Savings

obtain the respective tax shield values

$$VTS^{NC} = \tau p(Int) \quad (3.3.6)$$

and

$$VTS^C = \tau p(Int) - \tau p(C). \quad (3.3.7)$$

Both tax shield values include the price of interest payments, $p(Int)$. Equation (3.3.7) additionally comprises the price of the loss on principal payments, $p(C)$. To quantify the effect of the tax treatment of a COD, we perform the following steps: First, we regard the price of debt at t to find an expression for $p(Int)$. Second, by using the obtained expression we simplify the Equations (3.3.6) and (3.3.7) which is followed by analyzing the price of the cancelled amount $p(C)$.

The value of debt at time t is the sum of the prices of interest and principal payments considering a possible default. Incorporating the cancelled amount as in Equation (3.3.2) yields

$$D = p(Int) + p(PP) = p(Int) + p(D - C). \quad (3.3.8)$$

The firm issues debt at par so that D is at the same time value and principal payment. By noting that D is a certain quantity at time $t + 1$ and rearranging, we obtain the price of the interest payments

$$p(Int) = D - \frac{D}{R^f} + p(C) = \frac{r^f D}{R^f} + p(C). \quad (3.3.9)$$

Having found an expression for the interest payments in Equation (3.3.9) we restate the pricing equations (3.3.6) and (3.3.7) for both COD tax treatments to

$$VTS^{NC} = \tau p(Int) = \tau p(r^f D + C) = \tau \frac{r^f D}{R^f} + \tau p(C), \quad (3.3.10)$$

$$VTS^C = \tau p(Int) - \tau p(C) = \tau p(r^f D + C - C) = \tau \frac{r^f D}{R^f}. \quad (3.3.11)$$

The value decomposition of the cash flows to debtholders in interest and principal payments allows us to state our first result: In the case of a taxation of a COD, the terms $\tau p(C)$ cancel out.

3.3. Tax shield, tax treatment of a COD and proportional loss distribution

Independent of the riskiness of debt, the value of the tax savings is always determined as in the case for risk-free debt, i.e., the risk-free interest payments multiplied by the tax rate. As we observe from Equation (3.3.9), the interest payments compensate for the value of the losses on principal payments $p(C)$ above the value of risk-free interest payments $p(r^f D)$. Moreover, due to the (mathematical) cancellation of $p(C)$, Equation (3.3.11) is independent of the loss distribution between interest and principal payments or, stated in other words: Any assumption of losses on debt would lead to the same result for the value of tax savings. This case is equivalent to the case with debt being risk-free and consequently, the risk-free rate r^f would be paid on the principal D . For the case of no taxes on a COD, Equation (3.3.10) depends on $p(C)$ and is therefore sensitive to the loss distribution among interest and principal payments.

Notice that the analysis is similar for zero coupon bonds. In this case, the firm does not explicitly pay interest. Zero coupon bonds comprise an implicit interest payment which is the difference between the face value and the price. In the presented framework, the face value of a zero coupon bond is $D(1+r^c)$ and the price D . The implicit contractual interest payments amount to $r^c D$. Thus, the above analysis applies.⁶

3.3.2 Proportional loss distribution

In the preceding section, we found the pricing of the tax shield in case of no taxes on a COD to depend on the loss distribution among interest and principal payments. Subsequently, we consider the case of a proportional loss distribution and highlight which discount rate should be used within the interest tax shield valuation framework.

A proportional (or pro rata) loss distribution means that losses are distributed among interest and principal payments pro rata according to the respective promised payments. We represent the losses on interest by

$$L^{Int} = L \frac{r^c D}{(1+r^c)D} = L \frac{r^c}{1+r^c}, \quad (3.3.12)$$

and losses on principal payments, i.e., a cancellation of indebtedness, by

$$C = L \frac{D}{(1+r^c)D} = L \frac{1}{1+r^c}. \quad (3.3.13)$$

In order to find the pricing equation for the tax savings, we substitute Equation (3.3.12) into (3.3.1) and the combined terms in Equation (3.3.10):

$$VTS^{NC} = \tau p(r^c D - L \frac{r^c}{1+r^c}) = \frac{\tau r^c D}{1+r^c} p(1+r^c - \frac{L}{D}). \quad (3.3.14)$$

⁶For instance, IRS (IRS, 2016, p.13) refers to zero coupon bonds as discount debt instruments, where discounts are regarded as a form of interest.

3.3. Tax shield, tax treatment of a COD and proportional loss distribution

We use the representation for the promised yield, $r^c = r^f + R^f \frac{p(L)}{D}$ ⁷, i.e., $1 + r^c = R^f (1 + \frac{p(L)}{D})$, and observe that the term $p(1 + r^c - \frac{L}{D})$ equals one. Finally, using these preparations, we obtain for the tax shield value

$$VTS^{NC} = \frac{\tau r^c D}{1 + r^c}. \quad (3.3.15)$$

Our approach considers the general case of a proportional loss distribution with N states. Notice that a loss distribution is always proportional as long as no partial loss is possible. For instance Cooper and Nyborg (2008), regard a special case of our general approach where either the firm does not default or suffers a full default. We conclude that Equation (3.3.15) generally holds for no taxes on a COD and a proportional loss distribution.

To obtain Equation (3.3.15), we need the price relation

$$p(L^{Int}) = \frac{r^c}{1 + r^c} p(L) \quad (3.3.16)$$

to hold⁸. This is always true as soon as the losses on interest payments are represented by Equation (3.3.12). This might also hold for special cases of non-proportional loss distributions. We could redistribute losses L starting from the proportional solution $\frac{r^c}{1+r^c}L$ through adding a vector of cash flows ϵ with $p(\epsilon) = 0$ to obtain

$$p(L^{Int}) = \frac{r^c}{1 + r^c} p(L) = p\left(\frac{r^c}{1 + r^c} L + \epsilon\right), \quad (3.3.17)$$

under the constraint $0 \leq \frac{r^c}{1+r^c}L + \epsilon \leq \min(Dr^c, L) \leftrightarrow -\frac{r^c}{1+r^c}L \leq \epsilon \leq \min(r^c D, L) - \frac{r^c}{1+r^c}L$ for each state. Trivially, ϵ must be non-zero in at least one state to change the loss distribution to interest payments. With positive state prices, the price of ϵ can be zero only if ϵ is smaller than zero in at least one state and greater than zero in at least one state. Thus, examples of loss distributions other than proportional ones can be constructed so that the derived valuation equation holds as shown above. But the steps taken to arrive at such a result, i.e., to construct a payoff ϵ with the properties outlined above and add it to the proportional loss distribution, have no practical value. Eventually, loss distributions that do not fulfill the mentioned conditions, i.e., that have $p(\epsilon) \neq 0$, cannot be properly valued with Equation (3.3.15).

⁷We determine the promised yield by using $D = p((1 + r^c)D - L) = \frac{(1+r^c)D}{R^f} - p(L)$, where we divide by D , multiply by R^f and rearrange for r^c to obtain $r^c = r^f + R^f \frac{p(L)}{D}$.

⁸To find Equation (3.3.16), note $VTS^{NC} = \tau p(Int) = \frac{\tau r^c D}{1+r^c} = \tau p(r^c D - L^{Int})$ and rearrange to $p(L^{Int}) = \frac{p(Dr^c)(1+r^c) - Dr^c}{1+r^c} = \frac{r^c}{1+r^c} (p(D)(1+r^c) - D) = \frac{r^c}{1+r^c} \left(\frac{D}{R^f} (1 + r^f + R^f \frac{p(L)}{D}) - D\right) = \frac{r^c}{1+r^c} p(L)$.

3.4 Tax-adjusted discount rates, tax treatment of a COD and proportional loss distribution

In most business valuation settings, appraisers use tax-adjusted discount rates rather than the APV approach. Hence, it is important to outline how the tax-adjusted discount rate is affected by the tax treatment of a COD and a proportional loss distribution. Moreover, we discuss the role of the “cost of debt”, i.e., the expected return on debt within these valuation equations.

We start the subsequent analysis by defining the discount rates as (conditional) expected returns. Tax-adjusted discount rates are used to discount the expected unlevered free cash flows $E[FCF]$ in order to determine the levered firm value by

$$V^L = \frac{E[FCF]}{E[R^{*,(\cdot)}]}, \quad (3.4.1)$$

where $E[R^{*,(\cdot)}]$ denotes the tax-adjusted discount rate and (\cdot) is again the placeholder for the case of the tax treatment of a COD.

There are two representations for discount rates as defined in Equation (3.4.1) that yield the levered firm value when applied on the unlevered free cash flows. First, and recently more often discussed, tax-adjusted discount rates (TADR) that are directly derived from the APV. Second, the standard-textbook WACC which is a straightforward derivation from Equation (3.2.1). For both, we discuss the combined effect of the tax treatment of a COD and a proportional loss distribution.

3.4.1 Tax-adjusted discount rates

In this section, we analyze the impact of the tax treatment of a COD on the TADR assuming a proportional loss distribution. We start with Equation (3.2.2) and divide by V^L to obtain $1 - \frac{VTS}{V^L} = \frac{V^U}{V^L}$. Using Equation (3.4.1) and the fact that $V^U = \frac{E[FCF]}{E[R^U]}$, where $E[R^U]$ is the discount rate for the unlevered return on equity, we obtain $1 - \frac{VTS}{V^L} = \frac{E[FCF]}{E[R^U]} \frac{E[R^*]}{E[FCF]}$. Notice that the $E[FCF]$ terms cancel out. Thus, the general representation for the TADR is

$$E[R^{*,(\cdot)}] = E[R^U] \left(1 - \frac{VTS}{V^L} \right), \quad (3.4.2)$$

or, by using the leverage⁹ ratio $l = \frac{D}{V^L}$,

$$E[R^{*,(\cdot)}] = E[R^U] \left(1 - l \frac{VTS}{D} \right). \quad (3.4.3)$$

According to Equation (3.4.3), the TADR depends on the tax shield value. By referring back to our results obtained in Section 3.3, we state the tax treatment-dependent relations:

⁹Notice that for a possible extension to a multiperiod setting the WACC needs the assumption of constant leverage l .

3.4. Tax-adjusted discount rates, tax treatment of a COD and proportional loss distribution

1. For the case of no taxes on a COD, we use Equation (3.3.15) and obtain

$$E[R^{*,NC}] = E[R^U] \left(1 - \frac{\tau r^c l}{1 + r^c} \right). \quad (3.4.4)$$

2. To find an expression for the case of taxes on a COD, we substitute Equation (3.3.11) into (3.4.3):

$$E[R^{*,C}] = E[R^U] \left(1 - \frac{\tau r^f l}{R^f} \right). \quad (3.4.5)$$

As stated above, in the case of taxes on a COD, the loss distribution has no effect on the tax shield value and therefore no effect on the TADR (Equation (3.4.5)) either. In contrast to this, the TADR with no taxes on a COD (Equation (3.4.4)) is a function of the promised yield and generally holds only for a proportional loss distribution.¹⁰

As mentioned by Sick (1990), the results of several authors (e.g., Modigliani and Miller, 1963; Myers, 1974; Miles and Ezzell, 1980) might be interpreted such that the interest tax savings should be discounted by the bond rate or the expected return on debt, $E[r^D]$, often referred to as cost of debt¹¹. Here, we examine this statement carefully and test whether a TADR equation comprising the cost of debt such as

$$E[R^*] = E[R^U] \left(1 - \frac{\tau l E[r^D]}{E[R^D]} \right) \quad (3.4.6)$$

holds for one of the two assumptions on the tax treatment of a COD. Therefore, in comparison to Equation (3.3.8) we use an alternative representation of the return on debt:

$$R^D = \frac{(1 + r^c)D - L}{D}. \quad (3.4.7)$$

Equation (3.4.7) maps the return on debt as a function of the promised debt payments considering a possible loss. Since the promised yield and the debt value are certain quantities, we determine the expected return on debt by

$$E[R^D] = 1 + r^c - \frac{E[L]}{D}. \quad (3.4.8)$$

As we observe from Equation (3.4.8), the expected return on debt depends on the expected value of the loss on debt. In case of *risk-free debt*, the loss on debt is zero and the promised yield as well as the expected return on debt are equal to the risk-free rate. In fact, there is no difference between the promised yield and the expected return on debt. In case of *risky debt*, the expected

¹⁰As a recent example, in appendix 4.5 we show that Equation (M&N19), a main result in Molnár and Nyborg (2013), leads back to Equation (3.4.2) when losses are proportionally distributed.

¹¹Sick (1990) also states that besides the expected return on debt, the rate could even be the promised yield. But there is no further argument under which conditions which rate applies.

3.4. Tax-adjusted discount rates, tax treatment of a COD and proportional loss distribution

loss is different from zero. Thus, the expected return on debt is always smaller than the promised yield by the term $\frac{E[L]}{D}$. In turn, the promised yield reflects the maximum the debtholders can earn since in some states the return will be lower. Therefore, in the case of risky debt one cannot simply replace the promised yield by the expected return on debt.

We continue our analysis by discussing our results with a prominent example of the literature on TADRs by relating the return on risky debt with systematic risk, i.e., risk that cannot be diversified away. In the simplest case, returns on debt are uncorrelated with systematic risk implying an equivalence between the expected return on debt and the risk-free rate. This is due to the fact that standard asset pricing models such as the CAPM assume that only systematic risk is priced and idiosyncratic risk is not, since it can be diversified away. Therefore, debt with only idiosyncratic risk is expected to return only the risk-free rate. In case of taxes on a COD (see Equation (3.4.5)) one could replace r^f by $E[r^D]$ and R^f by $E[R^D]$ because they would be equal in this case. For instance, Arzac and Glosten (2005) assume at the beginning of their analysis a debt issue without systematic risk and use the respective applicable rate in their Equation (A&G 13)¹². For debt carrying systematic risk they further state that one can use a rate that includes a risk premium instead of the risk-free rate. Such a rate will be generally understood as the expected return on debt $E[R^D]$. As discussed above, if we consider risky debt, this rate is smaller than the promised yield and with debt carrying systematic risk, expected returns on debt are not equal to the risk-free rate. However, Arzac and Glosten do not state any assumption on the tax treatment of a possible COD or about the loss distribution. If we refer to our obtained results for risky debt with or without the taxation of a COD, a replacement of the risk-free rate with the expected return on debt is not possible for debt with systematic risk. When we understand the promised yield as the rate intended by Arzac and Glosten in their Equation (A&G 13), they would implicitly switch assumptions from taxes on a COD to the case with no taxes on a COD. However, in this case, the general validity of Equation (A&G 13) would require the proportional loss distribution assumption. This allows us to state an important remark. When considering risky debt one has to be clear about what is meant with notions such as “cost of debt”, “borrowing costs” etc.

3.4.2 Textbook WACC-like discount rates

It is clear from the tax shield and tax-adjusted discount rate equations that the assumed tax treatment of a COD affects the pricing of levered firms. Thus, the standard WACC should be affected by the tax treatment of a COD as well because it is just another representation of the equations above. For analyzing the textbook WACC, we start with rearranging Equation (3.2.1).

¹²Equation (A&G 13) in Arzac and Glosten (2005) shows the tax shield pricing assuming a constant leverage and a multi-period analysis. To compare their result with our single-period results, one can just leave out the rho-terms in their equation to obtain Equation (3.3.11) in this paper. For a multi-period setting with a constant leverage notice that in Equation (A&G 13) needs to be adjusted to correctly map the tax treatment of a COD and the prioritization assumption. For taxes on a COD, r^D needs to be replaced by the risk-free rate. To correctly map the case of no taxes on a COD and a pro-rata loss distribution, r^D should be replaced by the promised yield.

3.4. Tax-adjusted discount rates, tax treatment of a COD and proportional loss distribution

Generally, the levered free cash flows equal the sum of the unlevered cash flows and tax savings, or algebraically $FCF^L = FCF + TS$ (see e.g., Kruschwitz and Löffler, 2006). We further define equity cash flows as the residual of levered free cash flows after debtholders have received their payments: $ECF = FCF^L - Int - PP$. Thus, levered free cash flows comprise the flows to equityholders and to debtholders: $FCF^L = ECF + (Int + PP)$. Taking expectations and subtracting the tax savings yields the unlevered free cash flows $E[FCF] = E[ECF] + E[Int + PP] - E[TS]$. The cost of equity and debt are the expected returns on the respective cash flow. Thus, we can write $V^L E[R^*] = SE[R^E] + DE[R^D] - E[TS]$, divide by V^L and obtain the weighted average cost of capital

$$E[R^{*,(\cdot)}] = \frac{S}{V^L} E[R^E] + \frac{D}{V^L} E[R^D] - \frac{E[TS]}{V^L}, \quad (3.4.9)$$

where $E[R^E]$ is the expected return on levered equity. Using the leverage ratio $l = D/V^L$ and factoring out we rewrite the WACC equation to

$$E[R^{*,(\cdot)}] = (1 - l)E[R^E] + l \left(E[R^D] - \frac{E[TS]}{D} \right). \quad (3.4.10)$$

Assuming risk-free debt, the expected return on debt is equivalent to the risk-free rate, i.e., $E[R^D] = R_f$, and the tax savings are determined according to $E[TS] = \tau r_f D$. Having noted these equivalences for risk-free debt we obtain the standard textbook WACC¹³

$$E[R^{*,(\cdot)}] = (1 - l)E[R^E] + l(1 + r_f(1 - \tau)). \quad (3.4.11)$$

Although not apparent from this standard textbook equation, as shown in Equation (3.4.10), the WACC actually depends on the expected value of the tax savings, $E[TS]$. Hence, for the case of risky debt the tax treatment of a COD affects the WACC equation. Regarding the case that taxes are applicable on a COD, the tax savings depend on the corporate tax rate, the actually paid interest and the cancelled amount of debt, i.e., $TS^C = \tau(Int - C)$. Since interest payments can be restated in terms of promised interest payments, the loss on the overall debt issue and the cancelled amount, $Int = r^c D - (L - C)$, the tax savings can be rearranged to $TS^C = \tau(r^c D - L)$. Using the alternative statement of returns on debt (3.4.7), we rearrange Equation (3.4.10) to

$$E[R^{*,C}] = (1 - l)E[R^E] + l(1 + E[r^D](1 - \tau)). \quad (3.4.12)$$

The first term in Equation (3.4.12) is the levered cost of equity. The second term represents the after-tax cost of debt. Besides the well-known financing effects and the tax-deductibility of interest

¹³More commonly, the WACC is represented by net rates: $E[r^*] = (1 - l)E[r^E] + lr_f(1 - \tau)$.

3.5. A numerical example of the valuation bias for no taxes on a COD

payments, the last term additionally reflects the impact of the tax treatment of a COD.¹⁴

Now we focus on the case of no taxes on a COD. Here, we substitute the result for the tax savings subject to no taxes on a COD, $TS = \tau Int = \tau(Dr^c - L^{Int})$ into Equation (3.4.10) and obtain

$$E[R^{*,NC}] = (1-l)E[R^E] + l \left(1 + E[r^D](1-\tau) - \tau \frac{E[C]}{D} \right). \quad (3.4.13)$$

In addition to the levered cost of equity and the cost of debt, the last term in Equation (3.4.13) represents the cancelled amount of principal payments. As a direct consequence, the loss distribution among interest and principal payments has an effect on the final equation of the WACC. As in the previous sections, we assume a proportional loss distribution with $C = \frac{L}{1+r^c}$. We may write the second term in brackets in Equation (3.4.13) to

$$\begin{aligned} 1 + E[r^D](1-\tau) - \tau \frac{E[C]}{D} &= \frac{D + (Dr^c - E[L])(1-\tau) - \tau \frac{E[L]}{1+r^c}}{D} \\ &= \frac{D(1+r^c) - E[L] - \tau \left(Dr^c - E[L] \left(1 - \frac{1}{1+r^c} \right) \right)}{D} = E[R^D] - \tau r^c \left(\frac{1+r^c - \frac{E[L]}{D}}{1+r^c} \right) \\ &= E[R^D] \left(1 - \frac{\tau r^c}{1+r^c} \right). \end{aligned}$$

Equation (3.4.13), i.e., the WACC for no taxes on a COD, simplifies to

$$E[R^{*,NC}] = (1-l)E[R^E] + lE[R^D] \left(1 - \frac{\tau r^c}{1+r^c} \right). \quad (3.4.14)$$

Notice that the term $\frac{\tau r^c}{1+r^c}$ is equivalent to $\frac{VTS^{NC}}{D}$. Since the discount rates for all debt cash flows and for the tax savings are all the same with a proportional loss distribution (see Krause and Lahmann, 2015), we can find a simple adjustment term multiplied with the expected return on debt for the case of no taxes on a COD. Summarizing, for the two assumption combinations: (1.) taxes on a COD and an arbitrary loss distribution or (2.) no taxes on a COD and a proportional loss distribution, we derived expressions comprising the cost of debt.

3.5 A numerical example of the valuation bias for no taxes on a COD

The previous analysis reveals the pro-rata loss distribution to be an implicit assumption of the standard tax shield pricing equations and in turn of the TADRs. In practice, other prioritization schemes might be relevant for the specific valuation setting. For example, consider the case of interest prioritization in default. In such a situation, losses on debt are first assigned to principal payments. Only if losses exceed the principal amount, interest payments are affected. Algebraically,

¹⁴ Assuming no taxes on a COD Kruschwitz and Löffler (2006) find an equivalent result in terms of net rates and refer to it as textbook WACC formula.

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we may express the actually paid interests by $Int = r^c D - \max(0, r^c D - EBIT)$,¹⁵ where $r^c D$ represents the promised interest and the max-function triggers the losses. Only if the $EBIT$ is smaller than the contractually fixed payments, losses reduce the actually paid interest.

To show the differences for no taxes on a COD arising from an application of Equation (3.3.15) assuming a pro-rata loss distribution in contrast to a setting with an interest prioritization, we outline a numerical example below. We compute the values for an interest prioritization based upon Equation (3.3.6) using state prices and apply Equation (3.3.15) implicitly assuming a pro-rata loss distribution. For both calculations, we determine the promised yield based upon $r^c = r^f + R^f \frac{p(L)}{D}$ and assuming an interest prioritization. Thus, we use the correct yield rate for both valuations. Equation (3.3.6) yields the “non-biased” values strictly reflecting the premise of the outlined scenario, henceforth termed as VTS , and Equation (3.3.15), termed as $VTS(b)$, creates “biased” values implicitly assuming a proportional loss distribution.¹⁶

We regard a setting with 20 states. The state-dependent $EBIT$ values are evenly spaced from 5 to 10. The corporate tax rate is set to $\tau = 30\%$ and the risk-free rate to $r^f = 5\%$. We generate 20 state prices consistent to the determination of the risk-free rate with $p(1) = \frac{1}{1.05}$. The state prices decrease from state one (with an $EBIT$ of 5) to state 20 ($EBIT = 10$). The expected unlevered free cash flows are $E[FCF] = 5.25$ and the unlevered firm value $V^U = 4.13$. We numerically determine for increasing debt levels the tax shield values for no tax on COD, VTS^{NC} , based on the Equations (3.3.6) and (3.3.15), the promised yield, r^c , and the leverage, l .

Figure 3.1 depicts the tax shield values VTS^{NC} (dotted line) and $VTS^{NC}(b)$ (dashed line) depending on the leverage l . First, observe the standard result that both tax shield values increase with increasing leverage. As soon as debt becomes risky at a leverage of 82%, i.e., the firm defaults at least in one state, the tax shield values start to differ. The value for interest prioritization, VTS^{NC} , is always greater than the one based on Equation (3.3.15), $VTS^{NC}(b)$. Second, we show the percentage difference calculated as $\%Diff = \frac{VTS^{NC}(b) - VTS^{NC}}{VTS^{NC}}$. Note that the difference increases with leverage as well and at a leverage of 100% shows a substantial value of -33% , i.e., $VTS^{NC}(b)$ is 33% lower than VTS^{NC} .

In Figure 3.2, we numerically analyze the implied TADRs based upon the non-biased (VTS^{NC}) and biased ($VTS^{NC}(b)$) tax shield values as well as their differences in value. While Equation (3.4.4) enables us to directly calculate the biased TADRs implicitly assuming a pro-rata loss distribution, $E[r^{*,NC}(b)]$, we determine the non-biased TADR based upon VTS^{NC} using the basic equation for TADR (3.4.2). While the expected free cash flows, $E[FCF]$, and in turn the unlevered firm value, V^U , are invariant to the leverage, the tax shield value changes as shown in Figure 3.1

¹⁵We obtain $Int = r^c D - \max(0, r^c D - EBIT)$ by noting that, for interest prioritization, whenever $Dr^c \geq EBIT$ interest payments are equal to $EBIT$ they also equal the levered free cash flows FCF^L . Retain the levered free cash flows to be defined as $FCF^L = FCF + TS = EBIT(1 - \tau) + \tau Int$. Due to interest prioritization, if $FCF^L \leq Int$, all levered free cash flows are used to pay interest: $Int = EBIT(1 - \tau) + \tau Int$. Rearranging and cancelling out $1 - \tau$ leads to $Int = EBIT$. Thus, in any state with interest losses $Int = EBIT$ and $L^{Int} = r^c D - EBIT$. For further reference, see also Krause and Lahmann (2015)

¹⁶In the following, we denote the biased values for the TADRs and the WACC with an additional (b).

3.5. A numerical example of the valuation bias for no taxes on a COD

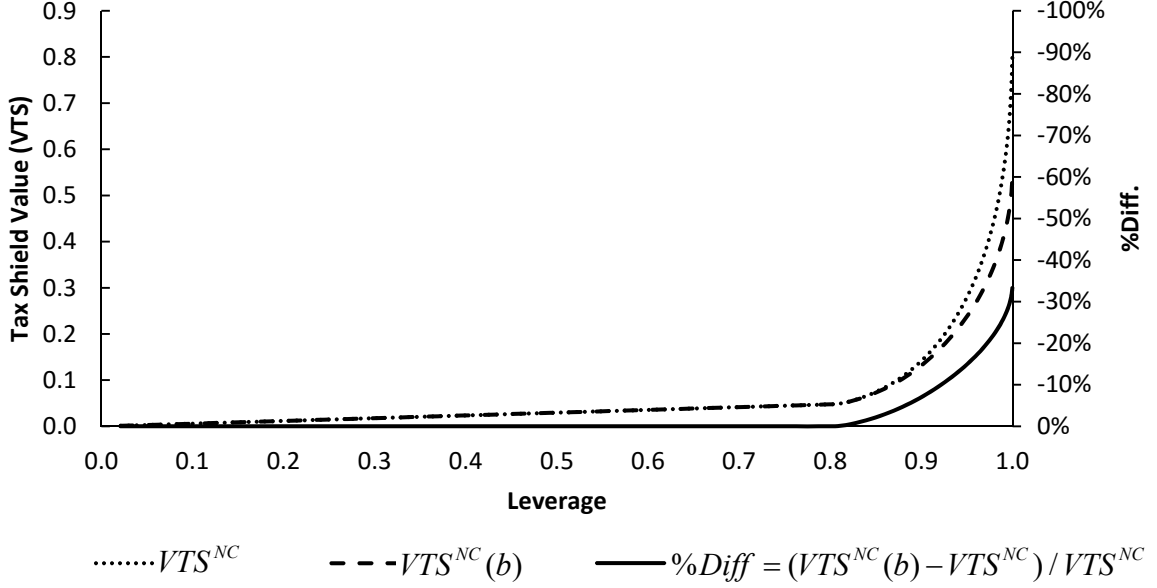


Figure 3.1: Tax shield values as a function of leverage, when interest prioritization is applicable.

The dotted and dashed line plot the tax shield value calculated by $VTS^{NC} = \tau p(Int)$ (dotted line) and $VTS^{NC}(b) = \frac{\tau r^c D}{1+r^c}$ (dashed line) at varying levels of leverage. The tax shield values are given on the left vertical axis. The percentage differences (full line) relative to the correct values are computed by $\%Diff = (VTS^{NC}(b) - VTS^{NC}) / VTS^{NC}$. The differences are mapped on the right vertical axis. Notice that the scale of the right axis is switched starting at zero and ending at negative 100%.

resulting in decreasing TADRs with increasing leverage. Figure 3.2 shows the non-biased TADRs for interest prioritization (dotted line), $E[r^{*,NC}]$, the biased TADRs (dashed line), $E[r^{*,NC}(b)]$, and their absolute difference (solid line), $Diff = E[r^{*,NC}] - E[r^{*,NC}(b)]$, as a function of leverage. For the relevant area, i.e., for risky debt, the biased TADR (dashed line) always exceeds the non-biased values (dotted line) because the tax shield value for a pro-rata loss distribution, $VTS^{NC}(b)$, is always smaller than the one for interest prioritization, VTS^{NC} . As above, the difference (solid line) has its maximum with 6.94% at a leverage of 100%. Figure 3.3 resembles similar results for the WACC. The correct WACC is the same as the correct TADR, as it should be since both are different ways of computing the same value. But the biased value for the WACC differs from the one for the TADR through the usage of the different pricing equations.

To further explain the differences between the curvature of the biased TADR (see Figure 3.2) and WACC (see Figure 3.3), we focus on the algebraic relation between the non-biased and biased values. Based on the Equations (3.4.3) and (3.4.4), determine the difference of the TADR by

$$E[R^{*,NC}] - E[R^{*,NC}(b)] = -lE[R^U] \left(\frac{VTS^{NC}}{D} - \frac{\tau r^c}{1+r^c} \right). \quad (3.5.1)$$

Using Equations (3.4.13) and (3.4.14) for the WACC, we compute the difference between the biased and non-biased WACC as

$$E[R^{*,NC}] - E[R^{*,NC}(b)] = -lE[R^D] \left(\frac{E[TS]/E[R^D]}{D} - \frac{\tau r^c}{1+r^c} \right). \quad (3.5.2)$$

3.5. A numerical example of the valuation bias for no taxes on a COD

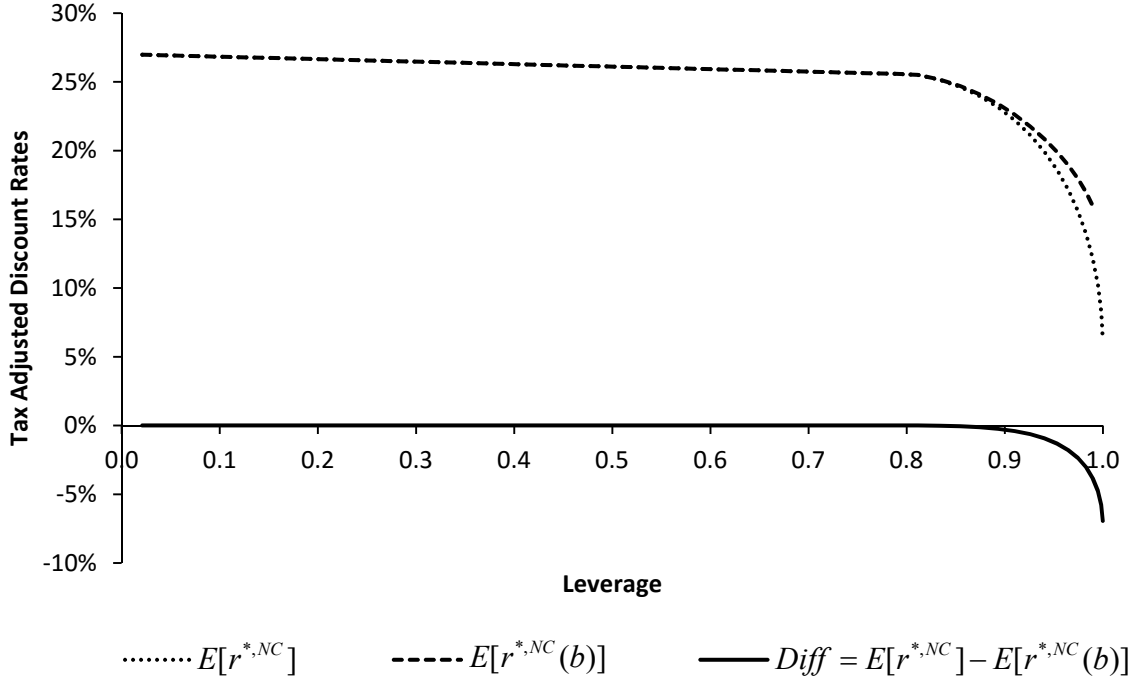


Figure 3.2: Biased and Non-biased TADR as a function of leverage, when interest prioritization is applicable.

The dotted and dashed line plot the non-biased ($E[r^{*,NC}]$) and biased TADRs ($E[r^{*,NC}(b)]$) according to Equation (3.4.2) and (3.4.4) at varying levels of leverage. The differences (full line) are computed by $Diff = E[r^{*,NC}] - E[r^{*,NC}(b)]$. Both, the values for the TADRs and the differences, are mapped on the left vertical axis.

Focus on the terms in brackets of both of the preceding Equations. Whenever the biased and non-biased calculations regard the case of a proportional loss distribution, the terms in brackets disappear and the difference vanishes. When losses are not proportionally distributed, the equations are not generally equal to zero and the result is also not generally the same. As for our example with interest prioritization, we regard the case of leverages when debt starts to become risky so that losses are attributed to principal payments but not yet to interest payments. Thus, the value of the tax shield is $VTS^{NC} = \tau r^c D / R^f$. Since the expected return on debt is greater than the risk-free rate: $E[R^D] > R^f$ implying $VTS^{NC} > E[TS] / E[R^D]$, we obtain the relation $\left(\frac{VTS^{NC}}{D} - \frac{\tau r^c}{1+r^c} \right) > \left(\frac{E[TS] / E[R^D]}{D} - \frac{\tau r^c}{1+r^c} \right)$ which explains the difference between the biased TADR and WACC. Furthermore, again for most practical cases and also in our example, the expected return on debt is less than the return on unlevered equity for any leverage ratio. In this case, the absolute value of the bias is smaller for the WACC than for the TADR.

From an economic perspective, the above results stem from the fact that with interest prioritization, fewer losses are assigned to interest payments than with a pro-rata loss distribution. For the latter, losses are assigned to both, principal and interest payments, from the very first loss while for an interest prioritization the principal amount needs to be exceeded by losses before interests are reduced. Thus, everything else equal, a pro-rata loss distribution creates smaller tax shields and in turn higher TADRs than a loss distribution with interest prioritization. Thus, applying

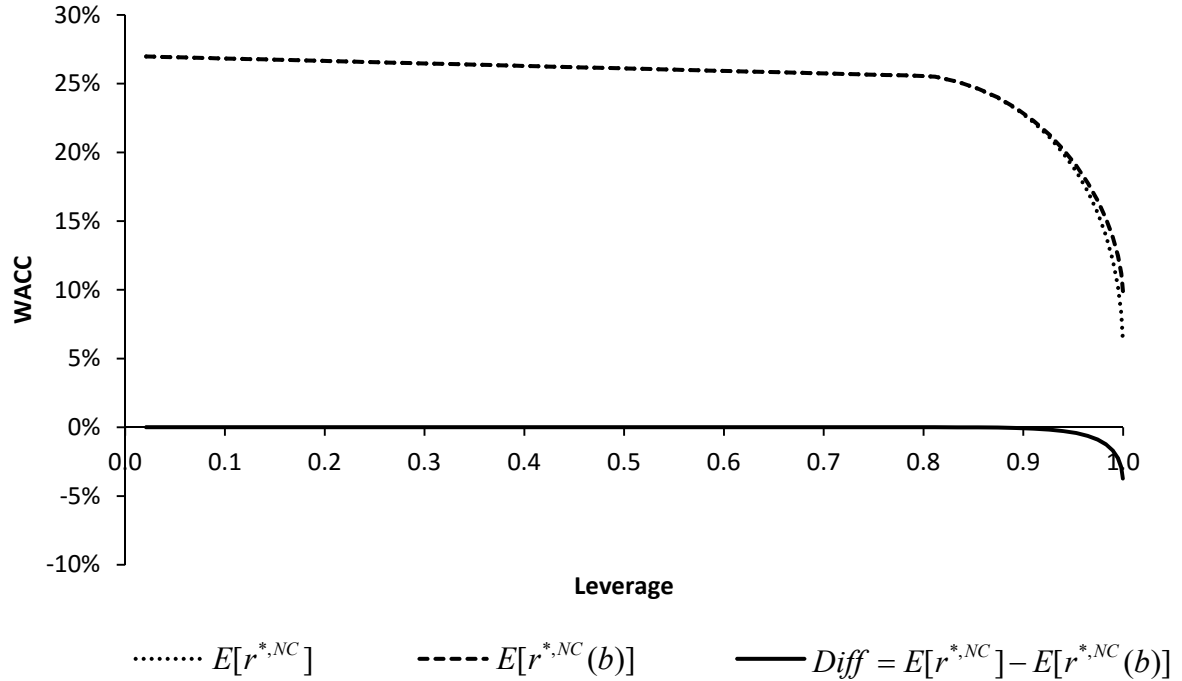


Figure 3.3: Biased and Non-biased WACC as a function of leverage, when interest prioritization is applicable.

The dotted and dashed line plot the non-biased ($E[r^{*,NC}]$) and biased WACC ($E[r^{*,NC}(b)]$) according to Equation (3.4.13) and (3.4.14) at varying levels of leverage. The differences (full line) are computed by $Diff = E[r^{*,NC}] - E[r^{*,NC}(b)]$. Both, the values for the WACC and the differences, are mapped on the left vertical axis.

the above derived standard equations implicitly assuming a pro-rata loss distribution results in a significant valuation bias when applied for other prioritization schemes.

3.6 Discussion and conclusion

Motivated by the fact that the tax treatment of a COD in case of a default implies possible negative effects on the tax shield, we aim at developing a simplified framework that illustrates the effect of taxes on a COD or their exemption in a concise style. Thereby, we abstain from stating manifold definitions regarding the pricing technique of certain cash flow streams. For our analysis it suffices to acknowledge that there is no arbitrage. Based on this, we show that the tax shield value is independent from any loss distribution between interest and principal payments for the case when taxes on a COD are paid. However, the tax shield value is not independent from the loss distribution when no taxes on a COD are paid.

We are able to show that a taxation of a COD renders the tax shield value equivalent to the value of a tax shield for risk-free debt at the same debt level. This value equivalence does not mean that tax savings become risk-free, but only that the pricing implication reflects the case of otherwise risk-free debt.

While this effect has been already shown by Sick (1990) and discussed for a special case by

3.6. Discussion and conclusion

Cooper and Nyborg (2008), we show that it stems from the risky components in the pricing equation of tax savings cancelling out when taxes on a COD are paid. For the case of no taxes on a COD, the tax-adjusted discount rate depends on the loss distribution among interest and principal payments. In contrast to Cooper and Nyborg (2008) with a two-state model and one state with a complete default, we regard the more general case of a proportional loss distribution.

We derive tax-adjusted discount rates for the case with taxes on a COD, where the loss distribution does not matter, and for the case without taxes on a COD, where we assume a proportional loss distribution. The derived equation for the tax-adjusted discount rate is the same as the one for the case of taxes on a COD, whereas only the risk-free rate is replaced by the promised yield rate. The equation in the case of no taxes on a COD is only generally true for a proportional loss distribution. For example, for a sequential loss distribution between interest and principal payments as in Molnár and Nyborg (2013) and Krause and Lahmann (2015), the equation will regularly give incorrect results. Moreover, our findings suggest that the use of the “cost of debt”, i.e., the expected return on debt, as a parameter accounting for risky debt within the tax-adjusted discount rate is appropriate only when debt has no systematic risk and a COD is taxed, so that there is no difference between the risk-free rate and expected returns on debt. If debt includes systematic risk, the cost of debt or expected return on debt is greater than the risk-free rate and cannot be a parameter in the tax-adjusted discount rate. Thus, the usage of the cost of debt is very limited under this assumption. For the case of no taxes on a COD, we need the promised yield rate as a parameter in the tax-adjusted discount rate. The cost of debt is only equal to the promised yield rate for risk-free debt, which is not the issue here since we want to analyze risky debt.

Summarizing, when analyzing the impact of default risk on business valuation or specifically on the tax shield, it is important to be aware of inconsistencies within the derived framework. For the valuation practice, we highlighted a possible pitfall when faced with mapping the risk of default. Certainly, knowing the relation between the prerequisites and the valuation equations limits the possibility of biased valuations.

3.7 Appendix - Comparison of the TADR to recent literature

We compare Equation (3.4.4) to another recent result for TADR in Molnár and Nyborg (2013). One of their main results is formula (M&N 19), which also considers personal taxes, such as taxes on capital gains and interest income. To obtain comparability, we set all personal tax rates to zero. Their result differs from our expression (3.4.4). In order to highlight the differences, we depict their Equation (M&N 19) only incorporating corporate tax rates and adjust their notation by adding a prime to avoid notation conflicts:

$$R'_L = R'_U - \frac{L'Y'_D T'_C (1 + R'_U) ((1 - \delta')R'_F + \delta'Y'_D - \alpha')}{(1 + R'_F)(Y'_D - \alpha')}. \quad (3.7.1)$$

Restating Equation (3.7.1) in terms of our notation yields

$$E[R^{*NC}] = E[R^U] \left(1 - \frac{lr^c \tau ((1 - \delta)r^f + \delta r^c - r^c - \frac{L}{D})}{R^f \frac{L}{D}} \right), \quad (3.7.2)$$

where δ is defined as in Molnár and Nyborg (2013) as the share of promised interest Dr^c that is paid in the default state.¹⁷ We show that Equation (3.7.2) only takes the form of Equation (3.4.4) for a proportional loss distribution. For such a loss distribution the interest paid in the default state is $Dr^c \left(1 - \frac{L}{D(1+r^c)}\right) = \delta Dr^c$, where $\frac{L}{D(1+r^c)}$ is the share of loss relative to promised debt payments and $1 - \frac{L}{D(1+r^c)}$ is the share of promised debt payments that the debtholder can recover. This implies

$$\delta = 1 - \frac{L}{D(1+r^c)}. \quad (3.7.3)$$

We set Equation (3.4.4) equal to (3.7.2), rearrange for δ and check for the equivalence with (3.7.3). $E[R^U]$, l , and τ cancel out, leaving us with

$$\frac{1}{1+r^c} = \frac{(1-\delta)r^f + \delta r^c - r^c - \frac{L}{D}}{R^f \frac{L}{D}}. \quad (3.7.4)$$

We rearrange for δ and obtain

$$\delta = \frac{\frac{R^f L}{D(1+r^c)} - \frac{L}{D} + r^c - r^f}{r^c - r^f}, \quad (3.7.5)$$

which we simplify to

$$\delta = \frac{\frac{L}{D} \frac{R^f}{(1+r^c)} - \frac{L}{D}}{r^c - r^f} + 1 = 1 + \frac{\frac{L}{D} \frac{1+r^f-1-r^c}{(1+r^c)}}{r^c - r^f} = 1 - \frac{L}{D(1+r^c)}, \quad (3.7.6)$$

¹⁷The term $1 + \alpha'$ in Molnár and Nyborg (2013) is the return on debt in the default state: $1 + \alpha' = \frac{D(1+r^c)-L}{D} = 1 + r^c - \frac{L}{D}$, i.e., $\alpha' = r^c - \frac{L}{D}$.

3.7. Appendix - Comparison of the TADR to recent literature

which is equivalent to Equation (3.7.3), the delta for a proportional loss distribution. That means given a proportional loss distribution, Equation (M&N 19) leads to (3.4.4). Since in Molnár and Nyborg (2013) the loss happens only in one state, there cannot be another loss distribution that leads from (M&N 19) to (3.4.4) for partial losses. Particularly in this case, any other distribution must lead to another price of losses on interest payments. That means (M&N 19) only leads to (3.4.4) for a proportional loss distribution.

3.7. Appendix - Comparison of the TADR to recent literature

Chapter 4

De- and re-levering betas with risky debt revisited¹

¹This article is based on Krause (2018a) and used in this dissertation under the Creative Commons Attribution 4.0 International License.

Bringing this publication into context

This article transfers the results of the two preceding articles to a related field. For assets that are not actively traded, one may use beta factors of traded asset, with otherwise similar characteristics. However, the companies may show different leverages so that betas need to be de- and re-levered. Usually, the expected return equation from the mean-variance CAPM is used for this exercise. The usual assumption made is that debt is risk-free. Even when risky debt is assumed, all the issues related to tax shield valuation are usually simplified. I close this gap in that I use the results from the first two articles to obtain equations that can be used for de- and re-levering betas. An example shows that using equations that rely on incorrect assumptions may substantially bias a valuation. As opposed to the second article on the taxation of cancelled debt, I also include the case of principal prioritization in a default and a multi-period analysis.

Abstract

De- and re-levering betas is important to obtain discount rates for assets that are not publicly traded. A de- and re-levering procedure is around for the case of risk-free debt. The procedure for risky debt is much less clear even under very simplifying assumptions. In this paper, I concretize and extend the procedure for de- and re-levering of betas for companies with risky debt. I derive procedures for different assumptions on the taxation of a cancellation of debt (COD) and for different assumptions regarding the distribution of losses on interest and principal payments. With a tax on the COD I obtain known results. However, without taxes on a COD, the distribution of losses on interest and principal payments matters and equations differ markedly for different assumptions on the assignment of losses to interest and principal payments. Furthermore, using a procedure that does not fit the COD treatment is likely to lead to substantial deviations for de- and re-levered betas from their correct values.

Keywords: Default Risk, Tax Treatment of Default, Betas, Leverage

4.1 Introduction

I build on the work of Krause and Lahmann (2017) and use it to extend their analysis for a de- and re-levering procedure for equity betas. I include interest and principal prioritization as additional cases, and I also discuss the case of constant leverage and an infinite horizon.

The main objective and contribution of this paper is to show how betas can be de- and re-levered when debt is risky and when different assumptions on the treatment of a cancellation of debt (COD) are made. The usual equations on de- and re-levering betas with risky debt implicitly assume that a COD is taxed as for example in Arzac and Glosten (2005). But as Krause and Lahmann (2017) mention, there are cases in which a COD remains untaxed. Furthermore, the assumption of a taxed COD is often not formally stated. However, presented equations change significantly when the assumption of taxes on a COD is dropped and when debt is risky.

The pricing framework builds mainly on the findings of Modigliani and Miller (1958) and Modigliani and Miller (1963). Additionally, Miles and Ezzell (1980) and Miles and Ezzell (1985) derive risk-adjusted discount rates in a multiperiod setting with corporate taxes and with constant leverage. In the more recent literature, several authors include the taxation of a COD into their analyses of corporations with risky debt (see e.g., Kruschwitz and Löffler, 2006; Cooper and Nyborg, 2008; Blaufus and Hundsdorfer, 2008).

Eventually, the de- and re-levering procedure relies on an expected return equation from an asset pricing model. This model does not need to be the classic mean-variance CAPM. A more general framework that relies on a stochastic discount factor is presented in Cochrane (2005) and can also be used.

I continue to introduce the basic notation for the single-period case. I present the beta equation for risk-free debt. Continuing with risky debt, I divide into the case of a taxation of a COD and the one without a taxation of a COD. At the end of the next section, I summarize the equations, discuss them and give a short example on possible miscalculations through applying the incorrect equation. In section 4.3, I briefly discuss the infinite horizon case with constant leverage, which, under simplifying assumptions, is similar to the single-period case. Section 4.4 summarizes the paper.

4.2 Levered and unlevered beta in a single-period setting

4.2.1 On COD taxation

According to Schwartzman and Brandstetter (2015), a U.S. company's forgiven or cancelled amount of debt from a bankruptcy or insolvency has to be recognized as gross income.² The COD is included in taxable income. However, there are several important exceptions from the general rule, most importantly bankruptcy and insolvency exclusions.

²See also the publication of the IRS (IRS, 2012, p. 26).

4.2. Levered and unlevered beta in a single-period setting

For bankruptcy, which is defined as a case under Title 11 of the U.S. Code, discharges of indebtedness under "Chapter 11 reorganization, Chapter 7 liquidations, and Chapter 13 bankruptcy proceedings" (see Schwartzman and Brandstetter (2015)) are excluded from taxable COD income. The discharge has to be ordered by a court or approved by a court. The COD is not included in gross income, but so called tax attributes "certain losses, credits, and basis of property must be reduced by the amount of excluded income (but not below zero)" (IRS, 2012, p. 26).

Insolvency, defined as liabilities in excess of the company's market value at the time right before the discharge, is also excluded from COD income. Only the insolvent amount can be excluded, and, again, certain tax attributes must be reduced. The reduction of tax attributes leads to a partial postponement of the tax on a COD instead of a full forgiveness (IRS, 2012, p. 26 ff).

The items mentioned above can only give a general picture. For more information please refer to the documentation from the Internal Revenue Service, for example, to IRS (2012) or to Schwartzman and Brandstetter (2015). Following the prior literature on pricing with and without a COD taxation, I will separately analyze the two extreme cases, the one in which a COD is taxed whenever debt is cancelled and the one in which it is never taxed. More elaborate models can be developed upon what is provided here.

4.2.2 Basic equations

I start with a simple setting with two points in time as in Krause and Lahmann (2017). I use the same notation and assumptions as in the mentioned paper. The single-period analysis has the advantages that additional assumptions on what happens after default are not necessary and that a simple notation without time subscripts is sufficient. Using simplifying assumptions, the infinite horizon version does not differ from the single period results as will be shown later. The starting point of the single-period analysis is the identity of (expected) cash flows:

$$E[FCF^L] = E[ECF] + E[Int + PP] = E[FCF] + E[TS]. \quad (4.2.1)$$

Levered free cash flows FCF^L are equal to the sum of equity cash flows ECF , as well as debt cash flows, which, in turn, consist of interest payments Int and principal payments PP . Alternatively, levered free cash flows are equal to unlevered free cash flows FCF and tax savings TS . Taking expected values through the operator $E[\cdot]$ keeps the identity. Equation (4.2.1) can be restated using values and returns

$$S \times E[R^E] + D \times E[R^D] = (S + D - VTS) \times E[R^U] + E[TS], \quad (4.2.2)$$

in which

$$V^U = V^L - VTS = S + D - VTS. \quad (4.2.3)$$

4.2. Levered and unlevered beta in a single-period setting

Here V^L is the value of the levered firm, V^U the value of the unlevered firm³, S the value of equity, D the value of debt, and VTS the value of tax savings. Furthermore, R^E is the return on (levered) equity, R^D is the return on debt, and R^U is the return on unlevered equity. In the expected value operator they are the respective expected returns. I use R for gross returns and r for net returns, where $R = 1 + r$. Values can be obtained by discounting expected cash flows, with $S = \frac{E[ECF]}{E[R^E]}$ for equity, $D = \frac{E[Int+PP]}{E[R^D]}$ for debt and $V^U = \frac{E[FCF]}{E[R^U]}$ for the value of the unlevered firm. Having defined the basic notation and relations, I turn to the risk-free case.

4.2.3 Risk-free debt and risk-free tax savings

I start with the simple but least relevant practical case for corporate valuations: risk-free debt. Since a COD will never happen with risk-free debt, its tax treatment does not play a role. I assume that the firm generates enough taxable income before interest payments to be able to fully deduct interest payments and to qualify for full interest tax savings. Then, all tax savings are equal to the risk-free interest payments times the tax rate on corporate profits τ : $TS = \tau \times Int = \tau \times r^f \times D$. Debt yields the risk-free interest rate r^f . I stress here that risk-free debt is a very special case, in which the firm is able to pay off all of its debt obligations in any future state. Leverage potentially affects the firms profitability so that, with higher leverage, risk-free debt becomes more and more unlikely. Compare Krause and Lahmann (2015) for a numerical example.

When debt is risk-free and the mean-variance CAPM is used to obtain expected returns, the following equation shows the relation of the levered and unlevered beta, i.e., of $\beta_{E,M}$ and $\beta_{U,M}$:

$$\beta_{E,M} = \left(1 + \frac{D}{S} \times \frac{1 + r^f \times (1 - \tau)}{R^f} \right) \times \beta_{U,M}. \quad (4.2.4)$$

Appendix 4.5 shows the derivation Equation (4.2.4). Equation (32) in Arzac and Glosten (2005) is similar to Equation (4.2.4) when their r^D , a discount rate for debt, is replaced by r^f . Their derivation is actually done in an infinite horizon setting. However, Equation (4.2.4) is the result of a single-period analysis. Adding periods does not add much to the analysis.

The term $\frac{1+r^f \times (1-\tau)}{R^f}$ is due to tax savings. With a zero tax rate this term equals one. For $r^f > 0$ and with the tax rate between zero and one (inclusive) this term is less than one. For $r^f < 0$ the term is greater than one.

Equation (4.2.4) also shows that higher debt relative to equity, i.e., a higher leverage scales up the levered beta. With higher leverage relatively more debt payments have to be made from the cash flows coming from the firm's assets. The remaining diminished cash flows go to equity holders. The return of those cash flows have a higher absolute value of the beta than before the increase in leverage. For positive unlevered beta the levered beta increases. If the unlevered beta is negative, the levered beta decreases. Appendix 4.6 shows how this occurs.

³The assets of the unlevered firm are the same as in the levered case, just the financing is different.

4.2. Levered and unlevered beta in a single-period setting

A higher tax rate decreases the levered beta. A higher tax rate increases tax savings and the value of them, which counters the effect of leverage, i.e., systematic risk is decreased through the presence of risk-free tax savings. Given that debt is risk-free, the risk-free tax savings are earned by equity holders. Therefore, tax savings increase the value of equity.

Equation (4.2.4) can also be stated in terms of leverage. This gives

$$\beta_{E,M} = \frac{1}{1-l} \times \left(1 - l \times \frac{\tau \times r^f}{R^f} \right) \times \beta_{U,M}. \quad (4.2.5)$$

A zero risk-free rate would also take out the tax effect. In this case interest payment would be zero so that there are no tax savings - just as for a zero tax rate:

$$\beta_{E,M} = \frac{1}{1-l} \times \beta_{U,M}. \quad (4.2.6)$$

After this introduction for risk-free debt, I turn to the more relevant case in the corporate world, the case of risky debt.

4.2.4 Risky debt

For risky debt the COD treatment is important. From Krause and Lahmann (2017), I restate the pricing equations for the value of the tax savings VTS for the different cases. Without (superscript NC) and with (superscript C) a tax on debt cancellation the tax savings are, respectively,

$$TS^{NC} = \tau \times Int \quad (4.2.7)$$

$$TS^C = \tau \times Int - \tau \times C. \quad (4.2.8)$$

Here τ is the deterministic tax rate on corporate profits, and C is the amount of debt that is cancelled, i.e., the losses on the principal D - the COD. When C is taxed, the whole tax savings are reduced by $\tau \times C$. Here it is assumed that $\tau \times C$ can actually be paid by the defaulting firm. Furthermore, interest payments can be expressed as

$$Int = r^c \times D - (L - C), \quad (4.2.9)$$

in which L are the total losses on debt, i.e., losses on interest L^{Int} and on principal payments C , and r^c is the coupon rate.⁴ According to Krause and Lahmann (2017), the values of the tax savings are then

⁴For a zero coupon bond it is the implicit coupon rate.

4.2. Levered and unlevered beta in a single-period setting

$$VTS^{NC} = \tau \times \frac{r^f \times D}{R^f} + \tau \times p(C) \quad (4.2.10)$$

$$VTS^C = \tau \times \frac{r^f \times D}{R^f}. \quad (4.2.11)$$

The term $p(C)$ is the price of the COD, i.e., $p(\cdot)$ is used as a pricing operator. The basis for further derivations is the identity

$$S \times E[R^E] + D \times E[R^D] = (S + D - VTS) \times E[R^U] + E[VTS]. \quad (4.2.12)$$

I continue with the case of taxation of a COD.

4.2.4.1 Taxes on debt cancellation

I use Equation (4.2.12) and substitute in the respective expression for the value of the tax shield and the tax savings from Equation (4.2.11) and Equation (4.2.8):

$$S \times E[R^E] + D \times E[R^D] = \left(S + D - \tau \times \frac{r^f \times D}{R^f} \right) \times E[R^U] + \tau \times (r^c \times D - E[L - C]) - \tau \times E[C]. \quad (4.2.13)$$

The $\tau \times E[C]$ terms cancel on the rhs and $r^c \times D - E[L] = D \times E[r^D]$ so that the equation turns to

$$S \times E[R^E] + D \times (1 + E[r^D] \times (1 - \tau)) = \left(S + D - \tau \times \frac{r^f \times D}{R^f} \right) \times E[R^U]. \quad (4.2.14)$$

I use the expected return equation for the mean-variance CAPM⁵ of the form

$$E[R^i] = R^f + \beta_{i,M} \times (E[R^M] - R^f) \quad (4.2.15)$$

for the expected return, in which i stands for the return on levered equity, on unlevered equity, on tax savings or on debt. The return R^M is the return on the market portfolio. I obtain

$$\begin{aligned} S \times (R^f + \beta_{E,M} \times (E[R^M] - R^f)) + D \times (1 + (R^f + \beta_{D,M} \times (E[R^M] - R^f) - 1) \times (1 - \tau)) \\ = \left(S + D - \tau \times \frac{r^f \times D}{R^f} \right) \times (R^f + \beta_{U,M} \times (E[R^M] - R^f)). \end{aligned} \quad (4.2.16)$$

⁵In (Cochrane, 2005, p.17 and p.19) a more general equation for an expected return equation using a stochastic discount factor is presented. In this case, the beta factor depends on how the stochastic discount factor is specified as a function of market data. The CAPM is a special case in which the stochastic discount factor is a linear function of the market return.

4.2. Levered and unlevered beta in a single-period setting

This expression can be simplified to

$$S \times \beta_{E,M} + D \times \beta_{D,M} \times (1 - \tau) = \left(S + D - \tau \times \frac{r^f \times D}{R^f} \right) \times \beta_{U,M}. \quad (4.2.17)$$

Rearranging yields

$$\beta_{E,M} = \left(1 + \frac{D}{S} \times \frac{1 + r^f \times (1 - \tau)}{R^f} \right) \times \beta_{U,M} - \beta_{D,M} \times \frac{D}{S} \times (1 - \tau). \quad (4.2.18)$$

When debt has no systematic risk, i.e., when $\beta_{D,M} = 0$, the equation reduces to the one with risk-free debt. Again, Arzac and Glosten (2005) have the same result in their equation (32). However, there is no reference to the treatment of debt cancellation. The following will show that the assumption of taxes on a COD is crucial to obtain this equation. Without it, valuations change and with them the de- and re-levering procedure. It is reasonable to expect the beta for debt to have the same sign as the one for equity. Furthermore, we usually expect positive equity betas together with positive debt betas. That means when the market is doing well as whole, debt contracts will also do better. When the market is not doing well, for example in recessions, there will be more defaults and the return on debt will also be lower. With a positive beta for debt, the additional debt-related term in Equation (4.2.18) reduces the levered beta, which counters the effect of the first term on the rhs of the equation, which increases the levered beta with more leverage.

4.2.4.2 No taxes on debt cancellation

Proportional loss distribution according to contractual debt payments. Krause and Lahmann (2017) derive an equation for the tax shield value for the case of a proportional loss distribution. Proportional loss distribution means that total losses L are distributed proportionally or pro rata according to the contractually agreed debt payments. Losses on interest and principal payments are, respectively, $L^{Int} = L \times \frac{r^c}{R^c}$ and $C = L \times \frac{1}{R^c}$. The equations for the value of the tax shield is

$$VTS^{NC} = \tau \times \frac{r^c \times D}{R^c}. \quad (4.2.19)$$

Tax savings are given by

$$TS = \tau \times \left(r^c \times D - L \times \frac{r^c}{R^c} \right). \quad (4.2.20)$$

I substitute both equations into Equation (4.2.12) and obtain:

$$S \times E[R^E] + D \times E[R^D] = \left(S + D - \tau \times \frac{r^c \times D}{R^c} \right) \times E[R^U] + \tau \times r^c \times D - \tau \times E[L] \times \frac{r^c}{R^c}. \quad (4.2.21)$$

4.2. Levered and unlevered beta in a single-period setting

I rewrite the term on the rhs: $\tau \times r^c \times D - \tau \times E[L] \times \frac{r^c}{R^c} = \tau \times r^c \times D \times \frac{E[R^D]}{R^c}$. I substitute this into the prior equation to obtain

$$S \times E[R^E] + D \times E[R^D] \times \left(1 - \frac{\tau \times r^c}{R^c}\right) = \left(S + D - \tau \times \frac{r^c \times D}{R^c}\right) \times E[R^U]. \quad (4.2.22)$$

Now, I use the expected return equations for the mean-variance CAPM:

$$\begin{aligned} S \times (R^f + \beta_{E,M} \times (E[R^M] - R^f)) + D \times (R^f + \beta_{D,M} \times (E[R^M] - R^f)) \times \left(1 - \frac{\tau \times r^c}{R^c}\right) \\ = \left(S + D - \tau \times \frac{r^c \times D}{R^c}\right) \times (R^f + \beta_{U,M} \times (E[R^M] - R^f)). \end{aligned} \quad (4.2.23)$$

This simplifies to

$$S \times \beta_{E,M} + D \times \beta_{D,M} \times \left(1 - \frac{\tau \times r^c}{R^c}\right) = \left(S + D - \tau \times \frac{r^c \times D}{R^c}\right) \times \beta_{U,M}. \quad (4.2.24)$$

Rearranging for the levered beta I obtain

$$\beta_{E,M} = \left(1 + \frac{D}{S} \times \frac{1 + r^c \times (1 - \tau)}{R^c}\right) \times \beta_{U,M} - \beta_{D,M} \times \frac{D}{S} \times \frac{1 + r^c \times (1 - \tau)}{R^c}. \quad (4.2.25)$$

or

$$\beta_{E,M} = \beta_{U,M} + (\beta_{U,M} - \beta_{D,M}) \times \frac{D}{S} \times \frac{1 + r^c \times (1 - \tau)}{R^c}. \quad (4.2.26)$$

In most cases, the beta for the unlevered firm is bigger than the one for debt because of priorities of debt cash flows to be paid to debt holders. That means $\beta_{U,M} - \beta_{D,M}$ is usually positive and with that the levered beta is greater than the unlevered beta - something that one would intuitively expect. In the less likely case, if $\beta_{U,M} < \beta_{D,M}$, then $\beta_{U,M} - \beta_{D,M} < 0$, and the levered beta is less than the unlevered beta.

Loss distribution not proportional to contractual debt payments. As Krause and Lahmann (2015) show, with a pro rata loss distribution according to contractual interest and principal payments, the expected rate of return on debt, i.e., the discount rate on debt $E[R^D]$, is equal to a weighted average of the expected rates of return on its components, i.e., the one on interest $E[R^{Int}]$ and the one on principal payments $E[R^{PP}]$. Notice that $p(Int) = \frac{E[Int]}{E[R^{Int}]}$ and $p(PP) = \frac{E[PP]}{E[R^{PP}]}$ define the discount rates for interest and principal payments. Krause and Lahmann (2015) show that, with interest or principal prioritization, expected rates of return on debt, interest and principal payments regularly differ. Without a COD taxation, tax savings are just interest payments scaled by the tax rate. Thus, the rate of return and the expected rate of

4.2. Levered and unlevered beta in a single-period setting

return on interest payments and on tax savings are equal: $E[R^{TS}] = E[R^{Int}]$.⁶ The expected return on debt as a weighted average of the expected returns on interest and principal payments is

$$\begin{aligned} E[R^D] &= \frac{E[Int] + E[PP]}{D} \\ &= E[R^{Int}] \times \frac{p(Int)}{D} + E[R^{PP}] \times \frac{p(PP)}{D}. \end{aligned} \quad (4.2.27)$$

Since $D = p(Int) + p(PP)$ and values are positive, the weights $\frac{p(Int)}{D}$ and $\frac{p(PP)}{D}$ are positive and add up to one. Due to this relation, possible relations of the three expected returns on debt cash flows are:

$$E[R^{TS}] = E[R^{Int}] < E[R^D] < E[R^{PP}] \quad (4.2.28)$$

$$E[R^{TS}] = E[R^{Int}] = E[R^D] = E[R^{PP}] \quad (4.2.29)$$

$$E[R^{TS}] = E[R^{Int}] > E[R^D] > E[R^{PP}]. \quad (4.2.30)$$

In the mean-variance CAPM, the only parameter that leads to different expected returns between different assets is the beta of an asset. The risk-free rate and the equity premium are do not depend on what kind of asset is regarded. Therefore, the respective betas must follow the same ordering as the expected returns⁷:

$$\beta_{TS,M} = \beta_{Int,M} < \beta_{D,M} < \beta_{PP,M} \quad (4.2.31)$$

$$\beta_{TS,M} = \beta_{Int,M} = \beta_{D,M} = \beta_{PP,M} \quad (4.2.32)$$

$$\beta_{TS,M} = \beta_{Int,M} > \beta_{D,M} > \beta_{PP,M}. \quad (4.2.33)$$

To derive equations for betas, I use Equation (4.2.12). I write it down in the form

$$S \times E[R^E] + D \times E[R^D] = (S + D - VTS) \times E[R^U] + VTS \times E[R^{TS}]. \quad (4.2.34)$$

Rearranging, simplyfying, and using the CAPM equations leads to

$$\beta_{E,M} = \frac{S + D}{S} \times \beta_{U,M} - \frac{D}{S} \times \beta_{D,M} + \frac{VTS}{S} \times (\beta_{TS,M} - \beta_{U,M}). \quad (4.2.35)$$

This equation allows the beta of the tax savings to be different than the one for total debt payments. In what follows, I establish equations that are comparable to the case with the pro rata loss

⁶This is true for a certain tax rate, which is assumed herein.

⁷This assumes a positive equity premium. However, a negative premium does not make sense for risk-averse investors.

distribution. I write the equation for the tax savings as

$$\begin{aligned}
 VTS &= p(TS) = \tau \times p(Int) \\
 &= \tau \times p(Dr^c - L^{Int}) \\
 &= \tau \times \left(\frac{Dr^c}{R^f} - p(L^{Int}) \right) \\
 &= \tau \times \frac{Dr^c}{R^c} \left(\frac{R^c}{R^f} - \frac{p(L^{Int})}{D} \times \frac{R^c}{r^c} \right) \\
 &= \tau \times \frac{Dr^c}{R^c} + \tau \times \left(p(L) \frac{r^c}{R^c} - p(L^{Int}) \right). \tag{4.2.36}
 \end{aligned}$$

The first term in the last equality is the equation for the value of the tax savings for a pro rata loss distribution according to contractual debt payments. In case of a pro rata loss distribution, the second term is always zero because then $p(L^{Int}) = p(L) \frac{R^c}{r^c}$. With loss distributions not proportional to contractual debt payments, the second term is usually not zero. Using that in Equation (4.2.35), I obtain

$$\begin{aligned}
 \beta_{E,M} &= \left(1 + \frac{D}{S} \times \frac{1 + r^c \times (1 - \tau)}{R^c} \right) \times \beta_{U,M} - \frac{D}{S} \times \beta_{D,M} + \frac{D}{S} \times \frac{\tau \times r^c}{R^c} \times \beta_{TS,M} \\
 &\quad + \frac{F}{S} \times (\beta_{TS,M} - \beta_{U,M}), \tag{4.2.37}
 \end{aligned}$$

with $F = \tau \times (p(L) \times \frac{r^c}{R^c} - p(L^{Int}))$. It turns out that additional information is needed. The beta of the returns on tax savings, i.e., on interest payments is needed as well as the price of losses on interest payments.⁸ I define $\beta_{\Delta TS,M} = \beta_{TS,M} - \beta_{D,M}$. I use this relation and restate Equation (4.2.37) as

$$\begin{aligned}
 \beta_{E,M} &= \left(1 + \frac{D}{S} \times \frac{1 + r^c \times (1 - \tau)}{R^c} \right) \times \beta_{U,M} - \beta_{D,M} \times \frac{D}{S} \times \frac{1 + r^c \times (1 - \tau)}{R^c} \\
 &\quad + \left(\frac{D}{S} \times \frac{\tau \times r^c}{R^c} + \frac{F}{S} \right) \times \beta_{\Delta TS,M} + \frac{F}{S} \times (\beta_{D,M} - \beta_{U,M}). \tag{4.2.38}
 \end{aligned}$$

This allows for a better comparability with Equation (4.2.26), i.e., the equation for the pro rata loss distribution according to contractual principal and interest payments. I continue with two prominent cases of loss distributions: interest and principal prioritization.

Loss distribution not proportional to contractual debt payments - Interest prioritization. A reasonable non-proportional loss distribution is the case of interest prioritization. Interest prioritization means that principal payments will incur losses first. Only if losses are greater than principal payments, interest will incur losses as well. Relation (4.2.28) is usually what we expect in this case.⁹ I simplify further. I assume that interest payments will never incur

⁸The price of losses $p(L)$ can be stated in terms of the debt value, the risk-free rate and the coupon rate.

⁹The other two relations are possible as well but unlikely in practice. For further information refer to Krause and Lahmann (2015)

4.2. Levered and unlevered beta in a single-period setting

losses. This is a reasonable assumption as long as interest payments are small relative to principal payments, which is what we mainly observe in practice. Under this assumption, interest payments are risk-free so that $L^{Int} = 0$ in any state. The price of losses on interest payments must be zero as well. It follows that the beta of tax savings is zero. The equation for interest payments turns to

$$Int = r^c \times D. \quad (4.2.39)$$

and tax savings are

$$TS^{NC} = \tau \times r^c \times D. \quad (4.2.40)$$

I discount this risk-free quantity at the risk-free rate, i.e., $E[R^{TS}] = R^f$, to obtain the value of the tax savings

$$VTS^{NC} = \tau \times \frac{r^c \times D}{R^f}. \quad (4.2.41)$$

Equation (4.2.37) condenses to

$$\beta_{E,M} = \left(1 + \frac{D}{S} \times \frac{R^f - \tau \times r^c}{R^f}\right) \times \beta_{U,M} - \beta_{D,M} \times \frac{D}{S}. \quad (4.2.42)$$

As for the pro rata distribution the levered beta is also a combination of the unlevered beta and a scalar as well as the debt beta and a scalar. However, the scalars differ here.

Loss distribution not proportional to contractual debt payments - Principal prioritization. The other extreme in terms of prioritization is principal payment prioritization. In this case losses are first assigned to interest payments. Only if losses are greater than interest payments, the excess amount of losses is assigned to principal payments. In this case a sensible assumption such as for interest prioritization is not available. Instead, I use Equation (4.2.38) and assumptions on F and the betas ¹⁰.

Given total losses on debt L , the losses on interest payments L^{Int} must be greater than the pro rata share of total losses: $L \frac{r^c}{R^c} < L^{Int}$. Since this is true for any state in which losses occur, the factor F is less than zero: $F = \tau \times (p(L) \frac{r^c}{R^c} - p(L^{Int})) < 0$. To parameterize, I assume that the price of losses on interest payments is equal to $p(L^{Int}) = p(L) (\frac{r^c}{R^c} + \alpha)$, with $\alpha \in (0, 1 - \frac{r^c}{R^c})$ as the percentage that the price of interest losses is higher than the pro rata share of the price of total losses. Using this parameter in the equation for F , I obtain

$$F = -\tau \times \alpha p(L) \quad (4.2.43)$$

$$= -\tau \times \alpha \times D \times \frac{r^c - r^f}{R^f}. \quad (4.2.44)$$

¹⁰For further discussion on principal prioritization see Krause and Lahmann (2015)

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The second equality follows from the equation for the coupon rate¹¹, which can be rearranged for the price of losses. Equation (4.2.38) turns to

$$\begin{aligned} \beta_{E,M} = & \left(1 + \frac{D}{S} \times \frac{1 + r^c \times (1 - \tau)}{R^c}\right) \times \beta_{U,M} - \beta_{D,M} \times \frac{D}{S} \times \frac{1 + r^c \times (1 - \tau)}{R^c} \\ & + \tau \times \frac{D}{S} \times \left(\frac{r^c}{R^c} - \alpha \times \frac{r^c - r^f}{R^f}\right) \times \beta_{\Delta TS,M} + \tau \times \alpha \times \frac{r^c - r^f}{R^f} \times \frac{D}{S} \times (\beta_{U,M} - \beta_{D,M}). \end{aligned} \quad (4.2.45)$$

Notice that with $\alpha \in (0, 1 - \frac{r^c}{R^c})$, it follows, for the term attached to $\beta_{\Delta TS,M}$, that $\frac{r^f}{R^f} < (\frac{r^c}{R^c} - \alpha \times \frac{r^c - r^f}{R^f}) < \frac{r^c}{R^c}$.¹² Furthermore, it is reasonable to assume that $\beta_{\Delta TS,M} > 0$, i.e., the beta of the returns on tax savings is greater than the one for the returns on debt. Since the return on tax savings and on interest payments are the same, the betas of the two figures are the same as well. With losses first assigned to interest payments, their returns will regularly have a higher beta than the beta for returns on principal payments and the one for returns on debt payments as whole.¹³

In the next subsection, I will compare the differences of the equations more in detail.

4.2.5 Overview and discussion

Table 5.1 shows an overview of the different cases treated in this paper. Some remarks are in order. For debt returns uncorrelated with market returns $\beta_{D,M} = 0$, the expected return on debt is equal to the risk free rate: $E[R^D] = R^f$. However, with risky debt, the coupon rate must be greater than the risk-free rate because the coupon rate must account for the default risk: $r^c > r^f$. This can also be restated as $r^c = r^f + \kappa$, in which $\kappa > 0$ is a credit risk premium. Thus, for $\beta_{D,M} = 0$, the equations for the risk-free case and for the risky case with taxes on a COD are the same. Krause and Lahmann (2017) found that the value of the tax savings for risky debt with taxed COD is the same as if debt is risk-free. However, without taxes on a COD the equations to adjust betas differ through the involvement of the coupon rate and possible other parameters.

A comparison of the equations in Table 5.1 makes most sense for the purpose of picking the correct equation given the case of the tax treatment and loss distribution. Then, one can evaluate the bias of picking an incorrect equation.

It does not make sense to pick a set of parameters and then try out all of the equations for the different cases intending to evaluate what the outcome would be given the parameters. The reason is that the different cases may be consistent only with different sets of inputs such as leverage and

¹¹The equation is $r^c = r^f + R^f \frac{p(L)}{D}$, which is just $D = p(D(1 + r^c) - L)$ rearranged for the coupon rate.

¹²Notice that for risky debt $\frac{r^f}{R^f} < \frac{r^c}{R^c}$. To see this multiply by R^f and R^c , which are both positive, and simplify. The result is $r^f < r^c$, which must hold for risky debt due to a positive credit risk premium.

¹³This is the likely case. Theoretically, it is also possible that this does not hold. The prioritization rules can be transformed into option-like payments. For a more detailed analysis of expected returns on options see Coval and Shumway (2001), and for an analysis with respect to tax savings see Krause and Lahmann (2015).

4.2. Levered and unlevered beta in a single-period setting

Table 4.1: Summary of cases and equations

| Risk | Tax | Loss distr. | Equation | | |
|-------|-----|-------------|---|-------------|---|
| None | NA | NA | $\beta_{E,M} = \left(1 + \frac{D}{S} \times \frac{1+r^f \times (1-\tau)}{R^f}\right) \times \beta_{U,M}$ | | |
| | | | Yes | Indifferent | $\beta_{E,M} = \left(1 + \frac{D}{S} \times \frac{1+r^f \times (1-\tau)}{R^f}\right) \times \beta_{U,M} - \beta_{D,M} \times \frac{D}{S} \times (1-\tau)$ |
| | | | | | Pro rata |
| Risky | No | Int. prio. | $\beta_{E,M} = \left(1 + \frac{D}{S} \times \frac{R^f - \tau \times r^c}{R^f}\right) \times \beta_{U,M} - \beta_{D,M} \times \frac{D}{S}$ | | |
| | | Pri. prio. | $\beta_{E,M} = \left(1 + \frac{D}{S} \times \frac{1+r^c \times (1-\tau)}{R^c}\right) \times \beta_{U,M} - \beta_{D,M} \times \frac{D}{S} \times \frac{1+r^c \times (1-\tau)}{R^c}$ $+ \tau \times \frac{D}{S} \times \left(\frac{r^c}{R^c} - \alpha \times \frac{r^c - r^f}{R^f}\right) \times \beta_{\Delta T S, M}$ $+ \tau \times \alpha \times \frac{r^c - r^f}{R^f} \times \frac{D}{S} \times (\beta_{U,M} - \beta_{D,M})$ | | |

The table shows the equations for de- and re-levering betas for the different cases. The column "Risk" indicates whether debt is risk-free or risky. The column "Tax" indicates whether a tax on a COD is paid or not. The column "Loss distr." indicates how total losses are assumed to be allocated to interest and principal payments.

Pro rata loss distribution means that total losses of debt are distributed to interest payments and principal payments according to contractual interest and principal payments. Interest prioritization means that all losses are first assigned to principal payments. The equation for interest prioritization presented here relies on the additional assumption that interest payments never incur losses. Principal prioritization means that all losses are first assigned to interest payments before principal payments are affected.

the coupon rate, because the tax assumption may influence the loss distribution.

I look at the factors scaling up and down the unlevered equity betas and the debt betas.

For the first three equations and the fifth equation notice that $1 + \frac{D}{S} \times \frac{1+r^f \times (1-\tau)}{R^f} > 1 + \frac{D}{S} \times \frac{1+r^c \times (1-\tau)}{R^c}$. To see that I rewrite the equations to $-\tau \times \frac{r^f}{R^f} > -\tau \times \frac{r^c}{R^c}$ and divide by $-\tau$ to obtain $\frac{r^f}{R^f} < \frac{r^c}{R^c}$. I multiply by R^f and R^c , which are both positive. That leads to $r^f \times (R^f + \kappa) < (r^f + \kappa) \times R^f$ which simplifies to $0 < \kappa$. The credit risk premium must be positive so that the inequality always holds.

Regarding equations three and four, the inequality $1 + \frac{D}{S} \times \frac{1+r^c \times (1-\tau)}{R^c} > 1 + \frac{D}{S} \times \frac{R^f - \tau \times r^c}{R^f}$ holds only for $r^c > 0$.

Regarding the first two equations and the fourth one, the inequality $1 + \frac{D}{S} \times \frac{1+r^f \times (1-\tau)}{R^f} > 1 + \frac{D}{S} \times \frac{1+r^c \times (1-\tau)}{R^c}$ always holds. This can be seen using the same simplifying steps as before.

I continue with the terms attached to the debt betas in equation two and three. For those equations $-\frac{D}{S} \times (1-\tau) > -\frac{D}{S} \times \frac{1+r^c \times (1-\tau)}{R^c}$. To see this, I simplify to $\tau > \tau \times \frac{r^c}{R^c}$, divide by τ and multiply by R^c to obtain $R^c > r^c$ or $1 > 0$, which always holds. Regarding equations three and four, for $r^c > 0$ the inequality $-\frac{D}{S} < -\frac{D}{S} \times \frac{1+r^c \times (1-\tau)}{R^c}$ holds.

Thus, for a positive coupon rate

$$1 + \frac{D}{S} \times \frac{1+r^f \times (1-\tau)}{R^f} > 1 + \frac{D}{S} \times \frac{1+r^c \times (1-\tau)}{R^c} > 1 + \frac{D}{S} \times \frac{R^f - \tau \times r^c}{R^f} \quad (4.2.46)$$

and

$$-\frac{D}{S} \times (1-\tau) > -\frac{D}{S} \times \frac{1+r^c \times (1-\tau)}{R^c} > -\frac{D}{S}. \quad (4.2.47)$$

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That means for a positive coupon rate and for positive betas for debt and unlevered equity, the levered beta will be smaller going down the equations in Table 5.1 until the fourth equation.

For positive betas debt betas may be larger than unlevered equity betas. For high debt betas and since the term including the debt beta is subtracted, the beta for levered equity may even be less than the one for unlevered equity, i.e., $\beta_{E,M} < \beta_{U,M}$. I will have a quick look at $\beta_{D,M}$ when $\beta_{E,M} = \beta_{U,M}$. For the second case, the case with a taxation of a COD, I obtain

$$\beta_{D,M} = \beta_{E,M} \frac{1 + r^f \times (1 - \tau)}{R^f(1 - \tau)}, \quad (4.2.48)$$

in which $\frac{1+r^f \times (1-\tau)}{R^f(1-\tau)} > 1$. For example for $r^f = 0.02$ and $\tau = 0.3$ the term is about 1.4 so that $\beta_{D,M}$ has to be more than 1.4 times bigger than $\beta_{E,M}$ to obtain $\beta_{E,M} < \beta_{U,M}$.

For the pro rata case the debt beta for $\beta_{E,M} = \beta_{U,M}$ is

$$\beta_{D,M} = \beta_{E,M}, \quad (4.2.49)$$

so that $\beta_{D,M}$ has to be greater than $\beta_{E,M}$ to obtain $\beta_{E,M} < \beta_{U,M}$.

For interest prioritization with the assumption of certain interest payments, I obtain

$$\beta_{D,M} = \beta_{E,M} \frac{R^f - \tau \times r^c}{R^f}, \quad (4.2.50)$$

in which $\frac{R^f - \tau \times r^c}{R^f} = 1 - \tau r^c / R^f$ less than one for $r^c > 0$. For example, for $r^f = 0.02$, $\tau = 0.3$ and $r^c = 0.06$ the term is 0.98. Thus, $\beta_{D,M}$ does not even have to be greater than the beta of unlevered equity for $\beta_{E,M} < \beta_{U,M}$.

The fifth equation treats principal prioritization. The equation requires the additional parameters α and $\beta_{\Delta TS,M}$. It was constructed so that the first line of the equation is the same as the equation for the pro rata case. Thus, the two additional terms in the equation for principal prioritization versus the equation for the pro rata case determine whether the resulting levered equity beta is greater, less than or equal to the levered equity beta for the pro rata distribution, when equal inputs are used. As was mentioned earlier, $\beta_{\Delta TS,M} > 0$ is a reasonable assumption for principal prioritization. Furthermore, for a positive risk-free rate the term $\tau \times \frac{D}{S} \times \left(\frac{r^c}{R^c} - \alpha \times \frac{r^c - r^f}{R^f} \right)$ is always positive. Assuming that $\beta_{U,M} > \beta_{D,M}$, makes the two additional terms positive. This leads to a levered equity beta that is greater than the one under the assumption of a pro rata distribution.

I provide a simple numerical example with the objective to observe the potential error from picking the incorrect procedure, i.e., the incorrect equation, given that the betas for unlevered equity and for debt are known. For the example, I choose the following parameters: $\tau = 0.3$, $l = 0.6$ (implying $D/S = 1.5$), $\beta_{U,M} = 0.9$, $\beta_{D,M} = 0.4$, $r^f = 0.02$, $\alpha = 0.2$, $\beta_{\Delta TS,M} = 0.3$ and $r^c = 0.06$. Table 5.2 shows that the betas for levered equity vary from 2.24 down to 1.63. That would imply expected returns for unlevered equity from 15.45 % down to 11.76 % for this

4.3. Levered and unlevered beta in an infinite period setting with constant leverage

example. The table shows that picking the incorrect equation has the potential for substantial errors in discount rates and thus for substantial misvaluations. Especially assuming risk-free debt when debt is actually risky has a big impact in this example because the additional term with the debt beta, which reduces the levered beta, is not used for risk-free debt. The assumption of a COD taxation also has a significant impact on the beta and the associated expected return. In turn, the figures presented without a COD taxation do not differ markedly. The reason is that for those equations and for a moderate risk-free rate as well as coupon rate, all the factors multiplied with $\beta_{E,M}$ and with $\beta_{D,M}$ do not differ a lot. Furthermore, the two additional terms in the equation for principal prioritization are small for reasonable parameters, such as the ones used in the example.

Table 4.2: Summary of cases and equations

| Risk | COD tax | Loss distrib. | $\beta_{E,M}$ | $E[R^E]$ |
|-----------|---------|-----------------|---------------|----------|
| Risk-free | NA | NA | 2.24 | 15.45% |
| | Yes | Does not matter | 1.82 | 12.93% |
| Risky | No | Pro rata | 1.64 | 11.82% |
| | | Interest prio. | 1.63 | 11.76% |
| | | Principal prio. | 1.65 | 11.87% |

4.3 Levered and unlevered beta in an infinite period setting with constant leverage

I take a simple approach and assume constant leverage and independently and identically distributed (i.i.d.) returns. At any point in time and in any state debt and equity cash flows have the same return distribution. The risk-free rate is assumed to be constant. The firm issues only single-period debt. It adjusts debt and equity at each point in time to keep leverage constant. Even after a default leverage is kept constant by whoever is the (new) owner of the company.

The returns are now defined as $R_{t+1}^E = \frac{ECF_{t+1} + S_{t+1}}{S_t}$ for the return on levered equity, $R_{t+1}^U = \frac{FCF_{t+1} + V_{t+1}^U}{V_t^U}$ for the return on unlevered equity, and $R_{t+1}^D = \frac{DCF_{t+1} + D_{t+1}}{D_t}$ for the return on debt.

The following identity must hold at any time:

$$ECF_{t+1} + S_{t+1} + DCF_{t+1} + D_{t+1} = FCF_{t+1} + V_{t+1}^U + TS_{t+1} + VTS_{t+1}. \quad (4.3.1)$$

The identity holds the same way taking conditional expectations:

$$\begin{aligned} E_t[ECF_{t+1}] + E_t[S_{t+1}] + E_t[DCF_{t+1}] + E_t[D_{t+1}] = \\ E_t[FCF_{t+1}] + E_t[V_{t+1}^U] + E_t[TS_{t+1}] + E_t[VTS_{t+1}]. \end{aligned} \quad (4.3.2)$$

4.4. Conclusion

I use the definitions of returns to obtain

$$S_t \times E_t[R_{t+1}^E] + D_t \times E_t[R_{t+1}^D] = (S_t + D_t - VTS_t) \times E_t[R_{t+1}^U] + E_t[TS_{t+1}] + E_t[VTS_{t+1}], \quad (4.3.3)$$

I represent the value of the tax savings as discounted next period's tax savings and discounted next period's value of all future tax savings, in which I assign two different discount rates to the respective figures:

$$VTS_t = \frac{E_t[TS_{t+1}]}{E_t[R_{t+1}^{TS}]} + \frac{E_t[VTS_{t+1}]}{E_t[R_{t+1}^{VTS}]}. \quad (4.3.4)$$

The first part, $\frac{E_t[TS_{t+1}]}{E_t[R_{t+1}^{TS}]}$, is the value of the tax savings of the next period. This value can be represented through the equations presented in the single-period model - only the time subscripts need to be adjusted. The second part, $\frac{E_t[VTS_{t+1}]}{E_t[R_{t+1}^{VTS}]}$, is the value at time t of all tax savings incurred after $t+1$. An important result of Miles and Ezzell (1985) is that for constant leverage the value of the tax shield is discounted at the discount rate for unlevered equity. Krause and Lahmann (2015) find that this is due to the tax shield value being proportional to the value of the unlevered firm when leverage is constant. They also find that the result holds in an i.i.d. setting with risky debt.

Since I assume constant leverage here, the equation for the tax shield value can be restated with $E_t[R_{t+1}^{VTS}] = E_t[R_{t+1}^U]$.

$$S_t \times E_t[R_{t+1}^E] + D_t \times E_t[R_{t+1}^D] = \left(S_t + D_t - \left[\frac{E_t[TS_{t+1}]}{E_t[R_{t+1}^{TS}]} + \frac{E_t[VTS_{t+1}]}{E_t[R_{t+1}^U]} \right] \right) \times E_t[R_{t+1}^U] + E_t[TS_{t+1}] + E_t[VTS_{t+1}], \quad (4.3.5)$$

Simplifying, the equation condenses to the same equation as for the single period (Equation (4.2.12)):

$$S_t \times E_t[R_{t+1}^E] + D_t \times E_t[R_{t+1}^D] = \left(S_t + D_t - \frac{E_t[TS_{t+1}]}{E_t[R_{t+1}^{TS}]} \right) \times E_t[R_{t+1}^U] + E_t[TS_{t+1}]. \quad (4.3.6)$$

Thus, the same equations for the de- and re-levering procedures of betas must follow for the infinite horizon case with constant leverage in an i.i.d. world. Equations of Table 5.1 can also be applied for this case.

4.4 Conclusion

I provide equations to re- and de-lever betas under different assumptions regarding the riskiness of debt and the taxation of a COD. When a COD is not taxed, the distribution of losses on interest and principal payments becomes important. I look at five cases: the risk-free case, the risky case with a taxed COD, the risky case without a taxed COD and pro rata distributed losses, the risky case without a taxed COD and interest prioritization, and the risky case without a taxed COD

4.4. Conclusion

and principal prioritization. I find that equations differ substantially so that the application of the incorrect procedure potentially leads to big errors in determining the discount rate. Additionally, the same equations for de- and re-levering betas for a single-period setting hold for a simple i.i.d. infinite horizon setting with constant leverage.

4.5 Appendix - Derivation of de- and re-levering of betas with risk-free debt in a single period setting

For risk-free debt

$$VTS = \tau \times \frac{r^f \times D}{R^f} \quad (4.5.1)$$

and $R^D = R^f$ as well as $TS = \tau \times D \times r^f$. Cash flows turn to

$$S \times E[R^E] + D \times R^f = \left(S + D - \tau \times \frac{r^f \times D}{R^f} \right) \times E[R^U] + \tau \times D \times r^f. \quad (4.5.2)$$

I use the expected return equation from the mean-variance CAPM for levered equity, i.e., $E[R^E] = R^f + \beta_{E,M} \times (E[R^M] - R^f)$, and for unlevered equity with $E[R^U] = R^f + \beta_{U,M} \times (E[R^M] - R^f)$.

I substitute both into Equation (4.5.2) to obtain

$$\begin{aligned} S \times (R^f + \beta_{E,M} \times (E[R^M] - R^f)) + D \times R^f = \\ \left(S + D - \tau \times \frac{r^f \times D}{R^f} \right) \times (R^f + \beta_{U,M} \times (E[R^M] - R^f)) + \tau \times D \times r^f. \end{aligned} \quad (4.5.3)$$

Simplification leads to

$$S \times \beta_{E,M} \times (E[R^M] - R^f) = \left(S + D - \tau \times \frac{r^f \times D}{R^f} \right) \beta_{U,M} \times (E[R^M] - R^f). \quad (4.5.4)$$

I divide by the equity premium and by S , which yields

$$\beta_{E,M} = \left(1 + \frac{D}{S} \times \frac{1 + r^f \times (1 - \tau)}{R^f} \right) \times \beta_{U,M}. \quad (4.5.5)$$

4.6 Appendix - Higher beta with higher leverage under risk-free debt

I add a prime symbol to the symbols of cash flows, values and returns after the debt increase. Assets remain the same. There are no taxes. I scale equity down by Δ so that $S' = S - S \times \Delta$, with $1 > \Delta > 0$, and debt increases accordingly so that $D' = D + \Delta \times S$. After the increase debt still remains risk-free. Equity cash flows change to $ECF' = ECF - \Delta \times S \times R^f$. That means a risk-free part of the equity cash flow goes away. It actually adds to debt cash flows. Debt cash flows change to $DCF' = (D + \Delta \times S) \times R^f$. The return on equity before the change was $R^E = ECF/S$. Now it turns to $R^{E'} = (ECF' - \Delta \times S \times R^f)/(S \times (1 - \Delta))$. Now, I look at the numerator of the equity beta which is a covariance of the form $Cov(R^E, R^M)$. The denominator is the variance of the return on the market portfolio $Var(R^M)$ and does not change with the change in leverage.

With the new return on equity the covariance turns to

$$Cov(R^{E'}, R^M) = Cov\left(\frac{ECF - \Delta \times S \times R^f}{S \times (1 - \Delta)}, R^M\right) \quad (4.6.1)$$

$$= Cov\left(\frac{ECF}{S \times (1 - \Delta)} - \frac{\Delta \times S \times R^f}{S \times (1 - \Delta)}, R^M\right) \quad (4.6.2)$$

$$= Cov\left(\frac{R^E}{(1 - \Delta)}, R^M\right) \quad (4.6.3)$$

$$= \frac{1}{1 - \Delta} \times Cov(R^E, R^M). \quad (4.6.4)$$

The term $1/(1 - \Delta)$ is greater one so that the covariance is scaled up. That means, with a positive beta, increasing leverage increases the beta. A negative beta would be even more negative. However, this case is very unlikely for equity. This case would be like an insurance against market risk.

4.6. Appendix - Higher beta with higher leverage under risk-free debt

Chapter 5

Effects of a capital gains tax on asset pricing¹

¹This article is based on Krause (2018b) and used in this dissertation under the Creative Commons Attribution 4.0 International License.

Bringing this publication into context

This paper departs from the view of a single firm that is to be valued, to issues of valuation of all assets in an economy. It is an extension of the work of Kruschwitz and Löffler (2009), who show cases when a capital gains tax in a mean-variance CAPM may lead to the same prices as if there were no capital gains taxes.

I go away from the mean-variance framework using stochastic discount factors and expected utility instead. I derive conditions for price equality in a tax and a no-tax economy. As in Kruschwitz and Löffler (2009) I look at the case of a zero risk-free rate. For that case I obtain the same portfolio rule as Kruschwitz and Löffler (2009). This rule leads to all agents consuming exactly the same in every state in both economies. I use their CARA setting in my expected utility setting and find that exponential utility with multivariate normal returns also leads to similar prices. As an additional case, I find that linear marginal utility also leaves prices unchanged. This is because with linear marginal utility discount factors of all agents are equal even in incomplete markets, and they are also equal to a discount factor of aggregate consumption. Aggregate consumption between the two economies is the same because in the tax economy all taxes are transferred back to the agents. It follows that the agents' individual discount factors must also be the same in the two economies.

Abstract

I extend and generalize the work of Kruschwitz and Löffler (2009). I find that, with a zero risk-free rate, the implicit price of capital gains tax payments is zero. I provide conditions in stochastic discount factor language when a capital gains tax has no effect on asset prices for the case of a zero risk-free rate. A sufficient condition for price equality with a zero risk-free rate is that agents consume the same in any state with and without taxes. Equilibria exist that guarantee equal consumptions, and they imply the same portfolio rules that Kruschwitz and Löffler (2009) find for the CAPM. Furthermore, for an exogenous non-zero risk-free rate, I show that exponential utility with multivariate normal payoffs, as well linear marginal utility leave prices unchanged. Equilibrium prices are independent of capital gains taxes in those cases. However, total wealth of agents is different between the tax and the no-tax economy.

Keywords: Capital gains tax, Stochastic discount factor, Portfolio theory, Constant absolute risk aversion, Linear marginal utility

5.1 Introduction

I build on the work of Kruschwitz and Löffler (2009) who assumed a single-period mean-variance capital asset pricing model (CAPM) with a flat tax on capital gains and tax transfers back to investors. They find that prices in a world with taxes on returns are the same as prices in a world without taxes if the risk-free rate is zero or investors have constant absolute risk aversion mean-variance utility. Instead of regarding a mean-variance CAPM as in Kruschwitz and Löffler (2009), I construct a model with agents that value expected utility over consumption, i.e., a consumption CAPM with heterogeneous agents. The fundamental results from Kruschwitz and Löffler (2009) also hold for such economies, but I find important extensions.

First, I look at economies with consumption at two times, and I examine the effect of the risk-free rate on asset pricing. I find that a non-zero risk-free rate leads to non-zero prices of tax payments. Even though not traded, prices of tax payments can be constructed from tradeable assets. For a zero risk-free rate capital gains taxes and the respective transfer payments have a zero (implicit) price. I construct two economies that have agents with equal endowments with shares of financial assets and consumption goods, equal utility functions and payoffs. I impose a tax on capital gains on one economy. I show that, for any tax economy, there is a no-tax economy with equal prices. This holds for two economies in which individual consumption of agents in one economy is the same as the consumption in the other economy in every state. Then, stochastic discount factors in the no-tax economy of any agent is the same as in the tax economy. Since taxes are not priced this leads to the same asset prices in both economies. Furthermore, I obtain the same portfolio rule as in Kruschwitz and Löffler (2009). This rule makes consumption profiles of investors equal in both economies with a zero risk-free rate. It follows that this rule is not just applicable to mean-variance CAPM economies but also to economies with expected utility maximizing agents, and in which a risk-free asset is traded and has a zero return. Without a zero risk-free rate price equality does not generally hold. For linear marginal utility it can be shown that it never holds.

I also regard the case of economies with consumption only in the future. In this case the risk-free rate is exogenous. For a zero risk-free rate price equality can be obtained again. For a non-zero risk-free rate, I show that exponential utility and multivariate normal payoffs lead to a pair of economies with equal prices. It is only necessary to pick equal prices of the risk-free assets in both economies. In contrast to Kruschwitz and Löffler (2009), who use mean-variance utility arguments, I use SDF arguments to derive this result. Furthermore, I show that aggregate wealth after initial consumption in the no-tax economy is different to the one in the tax economies - even though prices are the same. In the tax economy aggregate wealth is different from wealth in the no-tax economy by the price of aggregate transfer payments, which do not have zero prices as with a zero risk-free rate. The portfolio rule for risky assets is again the same as the one proposed in Kruschwitz and Löffler (2009). However, the rule for the risk-free asset differs. Furthermore, I find that utility functions that lead to marginal utility linear in consumption also lead to price equality.

The reason here is that individual pricing equations can easily be aggregated to a pricing equation that does not depend on the tax rate. With nonlinear marginal utility prices cannot generally be obtained since aggregation regularly does not lead to a pricing equation that is independent from the tax rate.

I contribute to the asset pricing literature that is especially concerned with tax effects on asset pricing. Much of the literature is concerned with the classic mean-variance CAPM such as Kruschwitz and Löffler (2009) and Eikseth and Lindset (2009), who consider tax transfers back to the investors. Sialm (2006), in turn, uses a representative agent model with an uncertain tax on consumption and tax transfers. He finds that aggregate consumption and therefore marginal utility growth is not affected when all taxes are transferred back. With certain and constant taxes there would not be an effect on asset prices versus no taxes. Brennan (1970) is a classic paper that incorporates various personal tax rates into the CAPM to arrive at pre-tax expected returns, but it does not consider transfers. Wiese (2007) builds on Brennan's work to develop a model that reflects the German tax code. I especially include SDF and consumption arguments into my analysis in the fashion of Cochrane (2014).

In Section 5.2, I introduce the basic economy without taxes and the economy with a flat and certain tax rate on capital gains. In the following Section 5.3, I show that for every no-tax economy there is a tax economy with equal asset prices in the cases when the risk-free rate is zero. When consumption only takes place in the future and therefore the risk-free rate is exogenous, equal prices are obtained when agents have exponential utility with normal consumption or when they have linear marginal utility. I continue to discuss the results and the limitations of the analyses in Section 5.4, where I also provide a simple numerical example. I conclude in Section 5.5.

5.2 The two basic economies

5.2.1 The basic finance economy without taxes

Payoff space. I model an endowment economy with financial assets. The economy exists at dates $t = 0$, when decisions are made and initial consumption takes place, and at $t = 1$, when payoffs are paid out and consumed. I add to the model of Kruschwitz and Löffler (2009) consumption at $t = 0$. I denote \mathbf{X}_r as an $N \times S$ matrix of tradeable, risky and elementary payoffs, in which N is the number of payoffs and S the number of possible states at $t = 1$. With elementary or basic payoffs, I mean non-redundant payoffs. Non-redundant, in turn, means that any single elementary payoff cannot be constructed through linear combinations of other payoffs. This matrix is augmented by a risk-free payoff X_0 , which is also non-redundant, so that $\mathbf{X} = (X_0 \ \mathbf{X}_r)'$ is an $(N + 1) \times S$ matrix of non-redundant payoffs. Thus, the payoff space is spanned by N elementary risky asset payoffs and a risk-free payoff. The number of states S can be greater than the number of assets so that an incomplete market is possible. I use the subscript s for individual states and the subscript j for the different financial assets so that the payoff j pays X_{js} in state s . To simplify notation, I

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put time subscripts only when necessary, such as for consumption, which is possible at $t = 0$ and at $t = 1$. I use all random variables as row vectors of dimension $1 \times S$. Constants such as prices of a single asset j , denoted p_j , can also be written as a $1 \times S$ vector of constant values.

Characterization of the agents and their maximization problems. There are $i = 1, \dots, I$ agents in the economy. Agents are rational and have the same complete set of information, i.e., they know the distributions of the payoffs. They are characterized through a time separable utility function $u_i(\cdot)$ over consumption and through initial (pre-trade) portfolio holdings $\bar{\mathbf{n}}_i$. At date $t = 1$ and in state s agent i consumes c_{is} units of a composite consumption good. One unit of a consumption good has a price of one at all times so that a payoff of one can buy exactly one unit of the consumption good. To address random variables such as agent i 's consumption or the j 'th payoff at $t = 1$, I leave out the subscript s for states and write c_{i1} and X_j , respectively. I denote \bar{c}_{i0} the endowment of agent i with consumption goods at time $t = 0$.

Agents maximize expected utility of consumption

$$\max_{c_{i0}, c_{i1}} E [\beta_i u_i(c_{i1})] + u_i(c_{i0}), \quad (5.2.1)$$

subject to the budget constraints at $t = 0$,

$$\bar{\mathbf{n}}_i' \mathbf{p} + \bar{c}_{i0} = \mathbf{n}'_i \mathbf{p} + c_{i0} \quad (5.2.2)$$

and at $t = 1$

$$c_{is} = \mathbf{n}'_i \mathbf{X}_s, \quad (5.2.3)$$

for $s = 1, \dots, S$. I use all collections of prices and asset weights as column vectors. I denote $E[\cdot]$ the expected value at time zero of its argument, $\mathbf{p} = (p_0 \ p_1 \ \dots \ p_j \ \dots \ p_N)'$ is the price vector of the $N + 1$ assets, $\mathbf{n}_i = (n_{i0} \ n_{i1} \ \dots \ n_{ij} \ \dots \ n_{iN})'$ is a vector of after-trade portfolio weights (I use $\bar{\mathbf{n}}_i$ for pre-trade portfolios), β_i the subjective time discount factor (or impatience factor), and $u_i(\cdot)$ the utility function. The expected value operator with a single random variable means a probability inner product. With a random variable z that means $E[z] = \sum_{s=1}^S \pi_s z_s$, in which π_s is the probability of state s . For prices I mostly use the short notation so that p_j is the price of a payoff X_j . When necessary, I also use prices as operators to make more clear what is priced, for example $p_j = p(X_j)$ is again the price of the payoff j . Furthermore, I use the subscript r to refer only to the risky assets $\mathbf{p}_r = (p_1 \ \dots \ p_j \ \dots \ p_N)'$ and $\mathbf{n}_{ir} = (n_{i1} \ \dots \ n_{ij} \ \dots \ n_{iN})'$, the subscript zero is related to the risk-free asset.² The utility function is differentiable and strictly monotonously increasing at a decreasing rate. Therefore, any additional unit of consumption adds to utility, and it is optimal to consume all of the payoffs, which justifies to write the budget constraints as

²Without too much abuse of notation, I also use the subscript zero for consumption at $t = 0$.

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equalities (Lengwiler, 2004, p.52). The equality of the budget constraints allows to substitute out consumption and to restate the maximization problem with respect to the portfolio weights and initial consumption.

The risk-free asset is in zero net supply $\sum_{i=1}^I n_{i0} = 0$. Therefore, I define a vector of aggregate asset holdings $\mathbf{n} = \sum_{i=1}^I \mathbf{n}_i$, which is $\mathbf{n}' = (0 \ 1 \ 1 \ \dots \ 1)$ because the risk-free asset is in zero net supply.

Equilibrium. The equilibrium is given through a vector of prices \mathbf{p} , consumption profiles c_{i0}, c_{i1} and portfolios \mathbf{n}_i for $i = 1, \dots, I$ so that each agent maximizes utility subject to his budget constraint, given prices \mathbf{p} . Furthermore, the market for the consumption good clears: $\sum_{i=1}^I \bar{c}_{i0} = \sum_{i=1}^I c_{i0}$ and $\sum_{i=1}^I c_{is} = \mathbf{n}' \mathbf{X}_s$ for $s = 1, \dots, S$. Financial assets are in positive net supply and markets clear so that $\sum_{i=1}^I n_{ij} = \sum_{i=1}^I \bar{n}_{ij} = 1$ for $j = 1, \dots, N$ and $\sum_{i=1}^I n_{i0} = 0$ for the risk-free asset. I assume that at least one equilibrium exists. Notice that equilibrium prices imply the absence of arbitrage opportunities (Lengwiler, 2004, p.50).

Pricing equations. I write the agent's optimization problem in terms of a Lagrangian

$$\mathcal{L}_i = E[\beta_i u_i(c_{i1})] + u_i(c_{i0}) - \lambda_i(\mathbf{n}'_i \mathbf{p} + c_{i0} - \bar{\mathbf{n}}'_i \mathbf{p} - \bar{c}_{i0}), \quad (5.2.4)$$

where λ_i is a Lagrange multiplier. I substitute in Equation (5.2.3) and take the partial derivatives with respect to portfolio weights and to initial consumption. Combining the results I obtain

$$\mathbf{p} = E \left[\mathbf{X} \beta_i \frac{u'_i(c_{i1})}{u'_i(c_{i0})} \right]. \quad (5.2.5)$$

I denote more compactly

$$m_i = \beta_i \frac{u'_i(c_{i1})}{u'_i(c_{i0})} \quad (5.2.6)$$

as agent i 's stochastic discount factor (SDF). Using this, I can price any single payoff X_j through:

$$p_j = E[m_i X_j]. \quad (5.2.7)$$

Here the expected value means that probabilities are induced to the inner product of X_j and m_i : $E[m_i X_j] = \sum_{s=1}^S \pi_s m_i X_{js}$. Through trading, agents find a price vector on which everyone agrees, i.e., $\mathbf{p} = E[m_i \mathbf{X}]$ for $i = 1, \dots, I$, and which maximizes utility.

In complete markets, \mathbf{X} is a square matrix with full rank, i.e., there are as many basic assets as states. The equation $\mathbf{p} = E[m_i \mathbf{X}]$ can be written as $\mathbf{p} = \mathbf{X} \kappa_i$, in which state prices are $\kappa_{is} = m_{is} \pi_s$ for $s = 1, \dots, S$ and π_s are objective probabilities of states s . When \mathbf{X} has full rank, there is a unique solution for κ_i . Since probabilities are objective probabilities, there is a unique SDF, i.e., every agent has the same SDF. It also follows that the state price vector can be expressed as a linear

5.2. The two basic economies

combination of basis assets and therefore lies in the payoff space. The same is true for the SDF.³

With incomplete markets, i.e., with $S > N + 1$, \mathbf{X} does not have full rank. The system of equations $\mathbf{p} = \mathbf{X}\kappa_i$ has less equations than unknowns so that there is more than one solution to the system. That means state prices and SDFs among agents may differ.

Pricing a risk-free payoff of one, I define the risk-free rate as $R_f = 1/E[m_i] = 1/p_0$ for $i = 1, \dots, I$. The term R_f is the gross risk-free rate: $R_f = 1 + r_f$. Thus, the pricing Equation (5.2.7) can be restated as

$$p_j = \frac{E(X_j)}{R_f} + Cov(m_i, X_j), \quad (5.2.8)$$

in which $Cov(m_i, X_j)$ is the covariance between the SDF and the payoff. As stated in Cochrane (2014), in incomplete markets the SDFs of agents m_i can differ and do not have to be within the payoff space. But there is one SDF m within the space of tradeable assets that prices all assets. This SDF is the probability induced projection of all of the agents' SDFs onto the payoff space. The relation between the unique SDF within the payoff space and any individual SDF is $m_i = m + \epsilon_i$, where ϵ_i is an error term orthogonal to the (probability induced) payoff space and therefore does not influence prices: $p_j = E(m_i X_j) = E((m + \epsilon_i)X_j) = E(mX_j)$, because $E(\epsilon_i X_j) = 0$ holds for all payoffs of the payoff space (Cochrane, 2005, p.66). The unique SDF within the payoff space can be used to price all payoffs but it will not necessarily lead to a possible portfolio rule for all agents, i.e., to a consumption profile that is within the payoff space. In complete markets the SDF is the same for every agent. In the standard CAPM, which does not require complete markets, the SDF is a linear combination of the market return: $m = a + bR^M$, where R^M is the return on the market portfolio and a and b are constants (Cochrane, 2005, p.152). In those two cases the single SDF leads straightforwardly to consumption rules within the payoff space.

5.2.2 The finance economy with taxes

I introduce another economy that has, compared to the no-tax economy, equal utility functions of agents $u_i(\cdot)$, equal impatience factors β_i , and an equal (pre-tax) distribution of payoffs of financial assets \mathbf{X} . The initial or pre-trade portfolios of agents with shares of assets are also the same, as well as the agents' perfect information about the payoff distributions. I introduce taxes on capital gains. To account for possible differences in prices, after-trade portfolios, and consumption profiles from the ones in the no-tax economy, I add an asterisk to them. Prices of taxed payoffs are denoted as $p_j^{*\tau} = p^*(X_j^\tau)$ and prices of pre-tax payoffs are denoted as $p_j^* = p^*(X_j)$.

Taxes. I define the tax base as the difference between the payoff and the price of the payoff: $X_{js} - p_j^{*\tau}$, in which $p_j^{*\tau}$ is the price of the after-tax payoff, i.e., of the payoff $X_{js}^\tau = X_{js} - T_{js} = X_{js} - \tau(X_{js} - p_j^{*\tau})$, in which T_{js} are taxes on the asset $j = 0, 1, \dots, N$ in states $s = 1, \dots, S$. Any

³For properties of the SDF under different assumptions such as market incompleteness see (Cochrane, 2005, pp.61-73).

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observed prices reflect possible tax effects. Investors consider the taxes they have to pay on the payoff when pricing the asset. I use $\tau \in (0, 1)$ as the tax rate and also as a superscript to denote after-tax figures when necessary. The tax rate is certain, constant, and the same for all agents. This is a simplification since tax rates can be observed to have an uncertain element and they often depend on certain characteristics of agents such as their income.⁴ Introducing an uncertain tax rate may introduce an additional covariance as well as an expectation into the pricing equation. An agent i pays capital gains taxes at the amount $T_{is} = \tau \sum_{j=0}^N n_{ij}^* (X_{js} - p_j^{*\tau}) = \tau \mathbf{n}_i^{*\prime} (\mathbf{X}_s - \mathbf{p}^{*\tau})$, and they receive transfer payments $Q_{is} = \tau \omega_i \mathbf{n}' (\mathbf{X}_s - \mathbf{p}^{*\tau})$ for $i = 1, \dots, I$, in which ω_i is the share of total tax revenues that is transferred to agent i with $\sum_{i=1}^I \omega_i = 1$. Transfer payments are predetermined amounts, i.e., they cannot be influenced by the agents. Positive and negative capital gains are taxed the same way. I discuss issues of this simplified tax system versus more realistic tax systems in Section 5.4. Aggregate tax payments are $T_s = \sum_{i=1}^I T_{is}$. They must be equal to aggregate transfer payments: $T_s = Q_s$. Individual transfer payments can also be written as $Q_{is} = \omega_i T_s$.⁵

The introduction of taxes and transfers does not introduce any new basic asset so that the payoff space is the same as in the no-tax economy. Any tax payment $T_{js} = X_{js} - \tau(X_{js} - p_j^{*\tau}) = X_{js}(1 - \tau) + \tau p_j^{*\tau}$ is just a linear combination of the pre-tax payoff X_j and a risk-free payoff.

Characterization of the agents and their maximization problems. Any agent maximizes expected utility of after-tax (and transfers) consumption

$$\max_{c_{i0}^*, c_{i1}^*} E[u_i(c_{i1}^*)] + u_i(c_{i0}^*), \quad (5.2.9)$$

subject to the budget constraints at $t = 0$

$$\bar{\mathbf{n}}_i' \mathbf{p}^{*\tau} + \bar{c}_{i0} = \mathbf{n}_i^{*\prime} \mathbf{p}^{*\tau} + c_{i0}^* \quad (5.2.10)$$

and at $t = 1$

$$c_{is}^* = \mathbf{n}_i^{*\prime} (\mathbf{X}_s - \tau(\mathbf{X}_s - \mathbf{p}^{*\tau})) + Q_{is}, \quad (5.2.11)$$

for $s = 1, \dots, S$. The variable \mathbf{n}_i^* is a vector of after-trade portfolio weights. I denote financial wealth that is left after initial consumption as $W_i^{*F\tau} = \mathbf{n}_i^{*\prime} \mathbf{p}^{*\tau}$ and total financial wealth after initial consumption, i.e., financial wealth including transfers as $W_i^{*F} = W_i^{*F\tau} + p^*(Q_i)$.

⁴See for example Sialm (2006) for a theoretical treatment of tax rate uncertainty on asset prices and Sialm (2009) for an econometric treatment.

⁵A case when agents receive transfers exactly at the amount they pay taxes is when $T_{is} = Q_{is}$ or $\tau \mathbf{n}_i^{*\prime} (\mathbf{X}_s - \mathbf{p}^{*\tau}) = \tau \omega_i \mathbf{n}' (\mathbf{X}_s - \mathbf{p}^{*\tau})$. This implies $(\mathbf{n}_i^{*\prime} - \omega_i \mathbf{n}') (\mathbf{X}_s - \mathbf{p}^{*\tau}) = 0$. Since $(\mathbf{X}_s - \mathbf{p}^{*\tau})$ includes risky assets, it cannot be a zero matrix. The vector $\mathbf{n}_i^{*\prime} - \omega_i \mathbf{n}'$ is a vector of zeros for $\mathbf{n}_i^{*\prime} = \omega_i \mathbf{n}'$, which is a very special case. With the risk-free asset in zero net supply this requires the first element of \mathbf{n}_i^* be zero and all of the remaining elements be equal to the constant ω_i . If the risk-free asset is in positive net supply, all elements of \mathbf{n}_i^* must be equal to ω_i .

Equilibrium. The equilibrium is given through a vector of prices $\mathbf{p}^{*\tau}$, consumption profiles c_{i0}^*, c_{i1}^* and portfolios \mathbf{n}_i^* for $i = 1, \dots, I$ so that each agent maximizes utility subject to his budget constraint, given prices $\mathbf{p}^{*\tau}$. Furthermore, the market for the consumption good clears: $\sum_{i=1}^I \bar{c}_{i0} = \sum_{i=1}^I c_{i0}^*$ and $\sum_{i=1}^I c_{is}^* = \mathbf{n}^{*\prime} \mathbf{X}_s$ for $s = 1, \dots, S$. That this holds comes from the fact that taxes are just redistributions and do not change aggregate values. Financial assets are in positive net supply and clear so that $\sum_{i=1}^I n_{ij}^* = \sum_{i=1}^I \bar{n}_{ij} = 1$ for $j = 1, \dots, N$, and $\sum_{i=1}^I n_{i0}^* = 0$ for the risk-free asset.

Pricing equations. The first order conditions lead to a similar pricing equation as for the no-tax economy, except that after-tax payoffs are priced:

$$p_j^{*\tau} = E[m_i^* X_j^\tau]. \quad (5.2.12)$$

The after-tax risk-free payoff is $X_0^\tau = X_0 - \tau(X_0 - p_0^{*\tau}) = 1 - \tau(1 - p_0^{*\tau})$, and the after-tax risk-free rate is

$$R_f^{*\tau} = \frac{1 - \tau(1 - p_0^{*\tau})}{E[m_i^*(1 - \tau(1 - p_0^{*\tau}))]} = \frac{1}{E[m_i^*]} = \frac{1 - \tau(1 - p_0^{*\tau})}{p_0^{*\tau}}. \quad (5.2.13)$$

The second equality follows from the fact that $1 - \tau(1 - p_0^{*\tau})$ is a constant, which can be taken out of the expectations in the denominator and therefore cancels with the term in the numerator. The third equality just restates that the denominator is actually the price of the cash flow $X_0^\tau = 1 - \tau(1 - p_0^{*\tau})$. The pre-tax risk-free rate is then

$$R_f^* = \frac{1}{E[m_i^*(1 - \tau(1 - p_0^{*\tau}))]} = \frac{1}{p_0^{*\tau}}. \quad (5.2.14)$$

Using $R_f^* = 1/p_0^{*\tau}$ the after-tax return can also be written as $R_f^{*\tau} = 1 + r_f^*(1 - \tau)$. If the risk-free rate is not taxed, it is $R_f^* = 1/E[m_i^*]$. Notice that since the risk-free asset is traded, every agent agrees upon the risk-free rate. It follows that the expected individual SDFs must be equal, which, in turn, are equal to the expected SDF within the payoff space: $E[m^*] = E[m_i^*]$ for $i = 1, \dots, I$.

In an economy with capital gains taxes, the expectations of the SDFs $E[m_i^*]$ play a special role. This is summarized in the following proposition.

Proposition 5.2.1. Assume an asset j with a pre-tax payoff X_j , and with an after-tax payoff X_j^τ with positive prices. Capital gains are taxed at a certain tax rate $\tau \in (0, 1)$. Assume further that $1/E[m_i^*] > \tau$. The prices of the pre-tax payoff p_j^* and of the after-tax payoff $p_j^{*\tau}$ are only equal as long as $E[m_i^*] = 1$ for $i = 1, \dots, I$. With $E[m_i^*]$ greater (less) than one the price of the after-tax payoff $p_j^{*\tau}$ is greater (less) than the price of the pre-tax payoff p_j^* .

Proof. After tax payoffs are defined as $X_j^\tau = (1 - \tau)X_j + \tau p_j^{*\tau}$. The respective price of this payoff

is

$$\begin{aligned} p_j^{*\tau} &= E[m_i^* X_j^\tau] = E[m_i^* ((1 - \tau)X_j + \tau p_j^{*\tau})] \\ &= (1 - \tau)E[m_i^* X_j] + \tau p_j^{*\tau} E[m_i^*]. \end{aligned} \quad (5.2.15)$$

This can be rewritten as

$$p_j^{*\tau} = p_j^*(1 - \tau) + \tau p_j^{*\tau} E[m_i^*], \quad (5.2.16)$$

which can be rearranged to

$$p_j^{*\tau} = \frac{p_j^*(1 - \tau)}{1 - \tau E[m_i^*]}. \quad (5.2.17)$$

Thus, when $E[m_i^*] = 1$, the tax terms cancel and prices of the pre-tax payoff and the one of the after-tax payoff are the same. In any other case the prices are not the same. Equation (5.2.17) shows further that for $E[m_i^*] > 1$, it follows that $(1 - \tau)/(1 - \tau E[m_i^*]) > 1$ so that $p_j^{*\tau} > p_j^*$ and vice versa. Equation (5.2.17) also shows that, given $E[m_i^*]$, i.e., the price of a payoff of one in every state, one can derive prices of pre-tax from after-tax payoffs and vice versa. The condition $1/E[m_i^*] > \tau$ ensures that the denominator of Equation (5.2.17) is positive. \square

I assume that $1/E[m_i^*] > \tau$ holds throughout the paper.

Notice that those pre-tax-after-tax price relations use an SDF of the tax economy m_i^* . Any relations to the SDFs of the no-tax economy, i.e., to m_i , are still to be obtained.

Notice also that $E[m_i^*] = 1$ implies that $E\left[\frac{u'_i(c_{i1}^*)}{u'_i(c_{i0}^*)}\right] = 1/\beta_i$. Expected growth of marginal utility of consumption is exactly equal to the inverse of the impatience factor. Higher growth implies a lower risk-free rate and lower growth a higher one. A simple log-normal model such as in Cochrane (2005, pp.10-12) allows for more interpretations of the risk-free rate in terms of consumption growth. In this case the risk-free rate is low when expected consumption growth is low or impatience is low, i.e., when beta is high.

The prior proposition has several implications.

Corollary 5.2.1. When the risk-free rate is not taxed, then, according to Equation (5.2.13), $R_f^* = 1/E[m_i^*]$, and it follows that $E[m_i^*] = 1$ and $r_f^* = 0$ are equivalent for all $i = 1, \dots, I$. Furthermore, $E[m_i^*]$ greater (less) than one is equivalent with the risk-free rate r_f^* being less (greater) than zero.

Corollary 5.2.2. When the risk-free rate is taxed, then, according to Equation (5.2.13), $R_f^{*\tau} = 1/E[m_i^*]$, and $E[m_i^*] = 1$ and $r_f^{*\tau} = 0$ are equivalent for all $i = 1, \dots, I$. Furthermore, $E[m_i^*]$ greater (less) than one is equivalent with the after-tax risk-free rate $r_f^{*\tau}$ being less (greater) than zero.

Corollary 5.2.3. In the case of a zero risk-free rate, the tax on capital gains has a zero price. From the above proposition follows that pre- and after-tax prices are the same, i.e., $p_j^{*\tau} = p_j^* - p^*(T_j) =$

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p_j^* , so that $p^*(T_j) = 0$. Furthermore, whether the risk-free rate of return is taxed as well does not matter when it is zero because taxes on that asset would also be zero.

In the following section, I continue to analyze equilibrium effects, i.e., how taxes affect prices and quantities in the no-tax and the tax economy.

5.3 Asset prices and portfolios in the no-tax and the tax economy

5.3.1 General conditions for price equality

I use the endowment economies, the one without and one with a tax on capital gains, that I have outlined in the prior section. I explore the general conditions under which prices are the same in the two economies.⁶

I continue to state the general conditions for asset prices be equal. I start with individual pricing equations and then continue with aggregate pricing equations and projections.

5.3.1.1 Individual pricing equations

Proposition 5.3.1. Asset prices in the no-tax and in the tax economy are equal, i.e., $\mathbf{p} = \mathbf{p}^{*\tau}$, if and only if

$$E[m_i \mathbf{X}] = E \left[m_i^* \mathbf{X} \frac{R_f^{*\tau}}{R_f^*} \right], \quad (5.3.1)$$

for $i = 1, \dots, I$.

Proof. I start with the vector of after-tax prices. Similar to the derivation of Equation (5.2.17) for a single price, the price vector is given by

$$\begin{aligned} \mathbf{p}^{*\tau} &= E[m_i^* \mathbf{X}^\tau] = E[m_i^* ((1 - \tau) \mathbf{X}^\tau + \tau \mathbf{p}^{*\tau})] \\ &= (1 - \tau) E[m_i^* \mathbf{X}] + \tau \mathbf{p}^{*\tau} E[m_i^*]. \end{aligned} \quad (5.3.2)$$

This can be rearranged to

$$\mathbf{p}^{*\tau} = \frac{1 - \tau}{1 - \tau E[m_i^*]} E[m_i^* \mathbf{X}]. \quad (5.3.3)$$

From Equations (5.2.13) we know that $R_f^{*\tau} = 1/E[m_i^*]$. Substituting that into the prior equation I obtain

$$\mathbf{p}^{*\tau} = \frac{1 - \tau}{1 - \tau/R_f^{*\tau}} E[m_i^* \mathbf{X}]. \quad (5.3.4)$$

⁶Notice that price equality concerns the tradeable financial assets. Transfer payments are not tradeable and do not belong to financial assets.

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I multiply the numerator and the denominator by $R_f^{*\tau}$, which yields

$$\mathbf{p}^{*\tau} = \frac{R_f^{*\tau}(1-\tau)}{R_f^{*\tau} - \tau} E[m_i^* \mathbf{X}]. \quad (5.3.5)$$

The denominator is $R_f^{*\tau} - \tau = 1 + r_f^*(1-\tau) - \tau = (1-\tau) + r_f^*(1-\tau) = R_f^*(1-\tau)$, so that the $1-\tau$ terms cancel. This leads to

$$\mathbf{p}^{*\tau} = E \left[m_i^* \mathbf{X} \frac{R_f^{*\tau}}{R_f^*} \right], \quad (5.3.6)$$

which I set equal to $\mathbf{p} = E[m_i \mathbf{X}]$ to obtain the condition in the proposition. \square

Corollary 5.3.1. From Equation (5.2.14), i.e., from the fact that $R_f^* = 1/p_0^{*\tau}$, and from $R_f = 1/p_0$ as well as from price equality of the risk-free assets follows that the risk-free rate in the no-tax economy is equal to the pre-tax risk-free rate in the tax economy: $R_f = R_f^*$. Furthermore, $R_f = R_f^*$ implies $E[m_i] = E[m_i^*]$ and vice versa, which follows from the definition of the risk-free rates.

For example from a pre-tax risk-free gross rate of return greater one, i.e., $R_f^* > 1$, follows that the after-tax rate is less than the pre-tax rate: $R_f^{*\tau} < R_f^* = R_f$. After accounting for taxes agents would require less return than they would in the no-tax economy. They value a unit payoff more than in the no-tax economy. For a zero risk-free rate pre- and after-tax rates are the same so that the valuation of a unit payoff would not change.

Corollary 5.3.2. Equation (5.3.1) can also be rewritten in terms of covariances:

$$\frac{E[\mathbf{X}]}{R_f} + Cov(m_i, \mathbf{X}) = \frac{E[\mathbf{X}]}{R_f^*} + Cov(m_i^*, \mathbf{X}) \frac{R_f^{*\tau}}{R_f^*}. \quad (5.3.7)$$

With $R_f = R_f^*$ from Corollary 5.3.1, I simplify to obtain

$$Cov(m_i, \mathbf{X}) = Cov(m_i^*, \mathbf{X}) \frac{R_f^{*\tau}}{R_f^*}, \quad (5.3.8)$$

for $i = 1, \dots, I$.

Furthermore, Proposition 5.3.1 implies a condition that guarantees that the proposition holds.

Corollary 5.3.3. The relation of the individual SDFs

$$m_i^* \frac{R_f^{*\tau}}{R_f^*} = m_i \quad (5.3.9)$$

for $i = 1, \dots, I$ is sufficient to obtain price equality for all assets.

This relation constitutes a strong assumption in that the SDF of any agent in the tax economy is proportional to the SDF of an equal agent in the no-tax economy in every state. I assume the agents' preferences to be the same in both economies so that a comparison makes sense. That

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means the agents' individual impatience factors and the parameters and functional form of their utility functions are the same. That also means it is consumption at $t = 0$ and consumption in the different states at $t = 1$ that determine the SDFs and possible differences in the SDFs of the two economies. Equation (5.3.9) can be restated as

$$u'_i(c_{i1}) = \zeta u'_i(c_{i0}^*), \quad (5.3.10)$$

in which $\zeta = u'_i(c_{i0})/u'_i(c_{i0}^*)R_f^{*\tau}/R_f^*$ is a constant that collects the ratio of the risk-free rates and the first derivatives of the utility functions of consumption at $t = 0$. This relation shows that Equation (5.3.9) implies that marginal utility at $t = 1$ be proportional.

With a zero risk-free rate the condition in Equation (5.3.1) simplifies to

$$E[m_i \mathbf{X}] = E[m_i^* \mathbf{X}], \quad (5.3.11)$$

and $m_i^* = m_i$ is sufficient to fulfill this condition, which is the same as condition (5.3.9) for a zero risk-free rate. Notice that the above conditions are derived from the price equations, which, in turn, are the rearranged first order conditions, i.e., the optimality conditions, of the agents. Thus making those equations hold guarantees optimality. Together they form an aggregate pricing equation.

5.3.1.2 Aggregate pricing equation

To obtain an aggregate demand function, I sum the individual equations of the form $\mathbf{p}^{*\tau} u'_i(c_{i0}^*) = E[\beta_i u'_i(c_{i1}^*) \mathbf{X}_r^\tau]$ over all agents. Rearranging for prices I obtain:

$$\mathbf{p}^{*\tau} = E \left[\frac{\sum_{i=1}^I \beta_i u'_i(c_{i1}^*) \mathbf{X}_r^\tau}{\sum_{i=1}^I u'_i(c_{i0}^*)} \right]. \quad (5.3.12)$$

The aggregate SDF is then

$$m^{a*} = \frac{\sum_{i=1}^I \beta_i u'_i(c_{i1}^*)}{\sum_{i=1}^I u'_i(c_{i0}^*)}. \quad (5.3.13)$$

This aggregate SDF prices all assets just as good as the individual SDFs. Given utility functions, it may help to find an aggregate pricing function.

Consumption, be it individual or aggregate, must lie within the payoff space. Even with taxes, when there are non-tradeable transfer payments, those payments can be replicated by tradeable payments because they are linear functions of tradeable payments. If marginal utility is linear in consumption, the quadratic utility case, all individual SDFs must lie within the payoff space. Since there can only be one SDF within the payoff space, all individual SDFs must be the same. Furthermore, it is well-known that this SDF can be written as a linear function in terms of aggregate consumption c_0 and c_1 , when all agents have the same time discount factor. Appendix 5.7 shows

a derivation. I come back to this important special case later.

5.3.1.3 Projections of SDFs

As pointed out in Section 5.2.1, there is a unique SDF within the payoff space that prices all assets, and which is related to the individual SDFs through $m = m_i + \epsilon_i$, with ϵ_i being an error term orthogonal to the probability induced payoff space (for the tax economy with an asterisk, respectively).

Proposition 5.3.2. Asset prices in the no-tax and in the tax economy are equal, i.e., $\mathbf{p} = \mathbf{p}^{*\tau}$, if and only if

$$E[m\mathbf{X}] = E\left[m^*\mathbf{X}\frac{R_f^{*\tau}}{R_f^*}\right], \quad (5.3.14)$$

in which m and m^* are the SDFs in the payoff space in the no-tax and the tax economy, respectively.

Proof. I use the relations $m = m_i + \epsilon_i$ and $m^* = m_i^* + \epsilon_i^*$ with errors orthogonal to the probability induced payoff space, i.e., $E[\epsilon_i\mathbf{X}] = 0$ and $E[\epsilon_i^*\mathbf{X}^\tau] = 0$ for all i . The price vector of the tax economy is

$$\begin{aligned} \mathbf{p}^{*\tau} &= E[m_i^*\mathbf{X}^\tau] = E[(m^* - \epsilon_i^*)\mathbf{X}^\tau] \\ &= E[m^*\mathbf{X}^\tau] - E[\epsilon_i^*\mathbf{X}^\tau] \\ &= E[m^*\mathbf{X}^\tau]. \end{aligned}$$

The term $E[\epsilon_i^*\mathbf{X}^\tau]$ is zero since the error term is orthogonal to the payoff space. For the no-tax economy the derivation is similar. The remainder is similar to the proof of Proposition 5.3.1. \square

Since the error terms do not affect the pricing of the assets, the corollaries follow just as before.

Corollary 5.3.4. Corollaries 5.3.1 to 5.3.3 also follow for the SDF within the payoff space, i.e., for m and m^* .

5.3.1.4 Budget constraints and market clearing

So far I have found a necessary and sufficient condition for price equality in Proposition 5.3.1 and a sufficient condition in Corollary 5.3.3. For an actual equilibrium allocation, budget constraints have to be met and markets need to clear as well. In the following, a tax and an equivalent no-tax economy will be compared, which are in equilibrium. Thus, apart from meeting conditions of price equality the budget constraints and market clearing need to hold, so that this step is also included in the following analyses.

I will continue as follows: Under the assumptions that the no-tax economy is in equilibrium, I will

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derive sufficient conditions for the existence of a tax equilibrium with prices equal to the ones in the no-tax economy. To do that, I will draw on the conditions established herein.

5.3.2 Economies with consumption at $t = 0$ and $t = 1$

5.3.2.1 A zero risk-free rate and equal consumptions in both economies

In the following I will show that, with a zero risk-free rate, for an equilibrium in the no-tax economy there exists an equilibrium in the tax economy in which agents circumvent redistribution through the capital gains tax and through the transfer payments using the same portfolio rule as in Kruschwitz and Löffler (2009). As Kruschwitz and Löffler (2009) point out, equilibria need not be unique so that other equilibria may exist that are not consistent with such an allocation.

From the prior section it is obvious that with a zero risk-free rate $R_f^{*\tau}/R_f^* = 1$. Then, the equality $m_i^* = m_i$ for $i = 1, \dots, I$ is sufficient to obtain price equality, since Corollary 5.3.3 is met. Since consumption at the different dates are the only variable arguments in the SDFs of the agents, it is clear that equal consumption of agents in both economies leads to equal SDFs. This follows from observation of Equation (5.3.10).

It remains to show that there is a portfolio rule that makes equal consumption possible. Budget constraints have to hold and markets have to clear. I show that the portfolio rule that ensures equal consumption is the same as the one in Kruschwitz and Löffler (2009).

Before I turn to the portfolio rule, I will make some remarks. With equal individual SDFs, i.e., with $m_i^* = m_i$ for $i = 1, \dots, I$, and with a zero risk-free rate, which makes prices of taxes and transfers zero, asset prices in both economies must be the same and pre-tax prices are equal to after-tax prices: $\mathbf{p} = \mathbf{p}^{*\tau} = \mathbf{p}^*$. Asset prices in the tax economy are $\mathbf{p}^{*\tau} = E[m_i^* \mathbf{X}^\tau] = E[m_i^* (\mathbf{X} - \mathbf{T})]$. Using this and noting that $E[m_i \mathbf{T}] = 0$, it follows that $\mathbf{p}^{*\tau} = E[m_i^* \mathbf{X}^\tau] = E[m_i^* (\mathbf{X} - \mathbf{T})] = E[m_i \mathbf{X}] = \mathbf{p}$. With equal initial portfolios equal prices imply that agents have the same financial wealth after initial consumption in both economies: $W_i^F = W_i^{*F\tau} = W_i^{*F}$.

Agents receive the same utility as in the no-tax economy. With a zero price of taxes the initial budget constraints of the agents are also equal to the ones of the no-tax economy. Thus, agents maximize utility and obey their budget constraints.

I continue to construct the portfolio rule so that consumption is equal in both economies and that markets clear. With equal initial portfolios and wealth, i.e., agents have the same initial characteristics in both economies, equal consumption means that an optimum in the no-tax economy is equivalent to an optimum in the tax economy.

Initial consumption is just a constant, which is set equal for any agent in both economies. Consumption at $t = 1$ needs more attention.

Proposition 5.3.3. Given equal prices in the tax and the no tax economy, consumptions at $t = 1$ of all agents $i = 1, \dots, I$ are the same in both economies if and only if risky portfolios of all agents

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i for the no-tax and the tax economy are related through

$$\mathbf{n}_{ir} = \mathbf{n}_{ri}^*(1 - \tau) + \omega_i \tau \mathbf{n}_r, \quad (5.3.15)$$

and weights on the risk-free assets are related through

$$n_{0i} = n_{0i}^*(1 - \tau) + \tau(W_i^F - \omega_i W^F). \quad (5.3.16)$$

Proof. Consumption of any agent i at $t = 1$ in the no-tax economy is simply $c_{i1} = \mathbf{n}'_i \mathbf{X}$. Consumption in the tax economy is

$$c_{i1}^* = \mathbf{n}'_i (\mathbf{X} - \tau(\mathbf{X} - \mathbf{p})) + \omega_i \tau \mathbf{n}' (\mathbf{X} - \mathbf{p}). \quad (5.3.17)$$

I use the no-tax price notation because $\mathbf{p} = \mathbf{p}^{*\tau}$ must hold for the portfolios that are implied. Every investor consumes the same in both economies if $c_{i1} = c_{i1}^*$ or

$$\mathbf{n}'_i \mathbf{X} = \mathbf{n}'_i (\mathbf{X} - \tau(\mathbf{X} - \mathbf{p})) + \omega_i \tau \mathbf{n}' (\mathbf{X} - \mathbf{p}). \quad (5.3.18)$$

Since $W_i^F = \mathbf{n}'_i \mathbf{p}$ and $W^F = \mathbf{n}' \mathbf{p}$, I restate the equation as

$$\mathbf{n}'_i \mathbf{X} = \mathbf{n}'_i (1 - \tau) \mathbf{X} + \tau W_i^F + \omega_i \tau \mathbf{n}' \mathbf{X} - \omega_i \tau W^F. \quad (5.3.19)$$

For complete markets given \mathbf{n}_i^* there is a unique solution for \mathbf{n}_i since \mathbf{X} is a square matrix of full rank. For incomplete markets the system of equations is overdetermined, i.e., a system with more equations (number of states) than unknowns (number of portfolio weights). Overdetermined systems need not have a perfect solution at all.⁷ However in this case there is a unique perfect solution, which will be verified below. I separate into risky and constant parts, which leads to

$$n_{i0} + \mathbf{n}'_{ir} \mathbf{X}_r = n_{i0}^*(1 - \tau) + \tau W_i^F - \omega_i \tau W^F + (\mathbf{n}'_{ir} (1 - \tau) + \omega_i \tau \mathbf{n}'_r) \mathbf{X}_r. \quad (5.3.20)$$

Now, simple observation shows that

$$\mathbf{n}'_{ir} = \mathbf{n}'_{ir}^*(1 - \tau) + \tau \omega_i \mathbf{n}'_r, \quad (5.3.21)$$

in which the vector \mathbf{n}'_r is the same as \mathbf{n}' without the first element, i.e., a vector of ones, and

$$n_{i0} = n_{i0}^*(1 - \tau) + \tau(W_i^F - \omega_i W^F) \quad (5.3.22)$$

is a solution to the system of equations. Systems of linear equations can have zero, one or infinitely

⁷One can still obtain an approximate solution in the least squares sense (see also Williams (1990)).

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many solutions. I found that there is at least one solution to this system of linear equations. It is also exactly one since the payoffs in the matrix \mathbf{X} are linearly independent, so that infinitely many solutions are not possible.

□

This is the same relation of shares of risky assets that Kruschwitz and Löffler (2009) propose for the mean-variance CAPM with taxes on capital gains, with transfers, and with a zero risk-free rate, to obtain equilibria at equal prices in a tax and a no-tax economy. Since I do not assume any specific utility function that would imply the mean-variance CAPM, I conclude that their proposition for portfolio weights is not limited to the mean-variance CAPM.

In the CAPM, I can further simplify because every investor holds the market portfolio⁸ so that all elements within the vectors \mathbf{n}_{ir} and \mathbf{n}_{ir}^* are equal, i.e., $n_{i1} = n_{i2} = \dots = n_{ij} = \dots = n_{iN}$ and $n_{i1}^* = n_{i2}^* = \dots = n_{ij}^* = \dots = n_{iN}^*$.

5.3.2.2 The case of linear marginal utility

A special case is marginal utility linear in consumption of all agents, i.e., something like $u'_i(c_{i1}) = a_i + b_i c_{i1}$, and equal time discount factors $\beta = \beta_i$ for all i . Then, all individual SDFs are equal and lie within the payoff space. With equal time discount factors, the SDF depends on aggregate consumption in $t = 0$ and $t = 1$ and some constants (see Appendix 5.7). In equilibrium agents consume all what they have since it is optimal to do that. Aggregate consumption must be the same in the no-tax and the tax economy, because agents are given the same endowments, and pre-tax payoffs are the same. Thus, for linear marginal utility and equal time discount factors the SDF(s) are the same in the no-tax and the tax economy. Given zero risk-free rates, asset prices must be the same as well. An example of this case for quadratic utility is given in Section 5.4.2.1. Furthermore, with equal SDFs in both economies, if the risk-free rate is not zero, there is no price equality, because Proposition 5.3.1 does not hold anymore.⁹

5.3.3 Economies with consumption only at $t = 1$

5.3.3.1 General remarks

I continue to look at economies that have no time zero consumption. Kruschwitz and Löffler (2009) limit their analysis to this kind of economies. In this case the risk-free rate is assumed to be exogenous to the economy. It is not the result of the trade-off of current and future consumption as in the model with consumption at $t = 0$ and at $t = 1$, because consumption at $t = 0$ does not

⁸Kruschwitz and Husmann (2012, pp.186-189) present the Tobin Separation Theorem together with the Mutual Fund Theorem, which state that every investor holds a share of the market portfolio and of the risk-free asset.

⁹With $m = m^*$, in which I leave out the subscript since all individual SDFs are the same, the equation in Proposition 5.3.1 turns into $E[m\mathbf{X}] = E\left[m\mathbf{X}\frac{R_f^{*t}}{R_f^*}\right]$, a statement which is not true for non-zero r_f^* .

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take place.¹⁰ I also simplify to assume that all of the agents have a time discount factor of one.

An agent's maximization problem is

$$\max_{c_{i1}^*} E[u_i(c_{i1}^*)], \quad (5.3.23)$$

subject to the budget constraints at $t = 0$

$$\bar{n}_{i0} p_0^{*\tau} + \bar{\mathbf{n}}'_{ir} \mathbf{p}_r^{*\tau} = n_{i0}^* p_0^{*\tau} + \mathbf{n}'_{ir} \mathbf{p}_r^{*\tau} \quad (5.3.24)$$

and at $t = 1$

$$c_{i1}^* = n_{i0}^* (X_0 - \tau(X_0 - p_0^{*\tau})) + \mathbf{n}'_{ir} (\mathbf{X}_r - \tau(\mathbf{X}_r - \mathbf{p}_r^{*\tau})) + \omega_i \tau \mathbf{n}'_r (\mathbf{X}_r - \mathbf{p}_r^{*\tau}). \quad (5.3.25)$$

As in Kruschwitz and Löffler (2009), I rearrange the time zero budget constraint for the quantity of the risky asset to obtain

$$n_{i0}^* = 1/p_0^{*\tau} (\bar{n}_{i0} p_0^{*\tau} + \bar{\mathbf{n}}'_{ir} \mathbf{p}_r^{*\tau} - \mathbf{n}'_{ir} \mathbf{p}_r^{*\tau}), \quad (5.3.26)$$

and substitute this expression into the one for consumption and solve the maximization problem to obtain:

$$E[u'_i(c_i^*) ((X_0(1-\tau) + \tau p_0^{*\tau}) \mathbf{p}_r^{*\tau} / p_0^{*\tau} (-1) + \mathbf{X}_r(1-\tau) + \tau \mathbf{p}_r^{*\tau})] = 0. \quad (5.3.27)$$

The equation can be restated as

$$E[u'_i(c_i^*) (\mathbf{X}_r^\tau - \mathbf{p}_r^{*\tau} R_f^{*\tau})] = 0 \quad (5.3.28)$$

so that rearrangement leads to

$$\mathbf{p}_r^{*\tau} = \frac{E[u'_i(c_i^*) \mathbf{X}_r^\tau]}{R_f^{*\tau} E[u'_i(c_i^*)]}, \quad (5.3.29)$$

with the SDFs $m_i^* = \frac{u'_i(c_i^*)}{R_f^{*\tau} E[u'_i(c_i^*)]}$. An apparent question is whether and in which cases $c_i = c_i^*$ for all i would lead to price equality. In this case the SDFs can be rewritten as $m_i^* = \frac{u'_i(c_i)}{R_f^{*\tau} E[u'_i(c_i)]}$. Multiplying by $R_f^{*\tau}/R_f^*$ gives $m_i^* \frac{R_f^{*\tau}}{R_f^*} = \frac{u'_i(c_i)}{R_f^* E[u'_i(c_i)]}$. SDFs in the no-tax economy are $m_i = \frac{u'_i(c_i)}{R_f E[u'_i(c_i)]}$. Since the risk-free rate is exogenous, I set $R_f^* = R_f$, i.e., $p_0^{*\tau} = p_0$ as it is done in Kruschwitz and Löffler (2009). Now the condition in Equation (5.3.9) holds and prices must be equal. Notice that with consumption only at $t = 1$, equal consumption does not lead to equal SDFs, but rather to

¹⁰The definitions in terms of prices of a pre- or after tax cash flow of one still hold: $R_f^{*\tau} = \frac{1-\tau(1-p_0^{*\tau})}{p_0^{*\tau}}$ and $R_f^* = \frac{1}{p_0^{*\tau}}$.

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proportional SDFs.

However, in equilibrium, the budget constraints have to hold as well. If and only if the portfolio rules derived in the prior part hold, will there be equal consumptions in both economies. It turns out that this only holds for a zero risk-free rate. To show this, I start with the budget constraint in the no-tax economy denoted as in Equation (5.3.26), and I substitute in Equation (5.3.15). This leads to

$$n_{0i} = \bar{n}_{0i} + (\bar{\mathbf{n}}'_{ir} - \mathbf{n}^*{}'_{ir}(1 - \tau) + \omega_i \tau \mathbf{n}'_r) \frac{\mathbf{p}_r}{p_0} \quad (5.3.30)$$

Now, I use the budget constraint $n^*_{0i} = \bar{n}_{0i} + (\mathbf{n}^*{}'_{ir} - \bar{\mathbf{n}}'_{ir})\mathbf{p}_r/p_0$ rearranged to $\bar{n}_{0i} = n^*_{0i} - (\mathbf{n}^*{}'_{ir} - \bar{\mathbf{n}}'_{ir})\mathbf{p}_r/p_0$ and $W^F = \mathbf{n}'_r \mathbf{p}_r$ and substitute both into the prior equation, which leads to

$$n_{0i} = n^*_{0i} + \tau(\mathbf{n}^*{}'_{ir} \mathbf{p}_r - \omega_i W^F) \frac{1}{p_0} \quad (5.3.31)$$

$$= n^*_{0i}(1 - \tau) + \tau(W^F_i - \omega_i W^F) \frac{1}{p_0}. \quad (5.3.32)$$

Notice that this is different from Equation (5.3.16) when p_0 is not one, i.e., when the risk-free rate is not zero. Thus, portfolio rules consistent with equal consumptions of agents cannot be obtained when prices are equal in both economies.

Even though this path is closed, there are some cases when price equality can be obtained. However, consumptions are not equal anymore. Notice that for a zero risk-free rate the equal consumption approach still leads to price equality the same way as in the model with initial consumption.

5.3.3.2 Multivariate normal payoffs and exponential utility

If the risk-free rate is not zero, condition (5.3.1) has to hold to make prices in the no-tax and the tax economy equal. Kruschwitz and Löffler (2009) discover that for the CAPM with constant absolute risk aversion (CARA), for every no-tax economy there is a tax economy with equal prices. They use arguments from a mean-variance utility approach. I use exponential utility, which is a CARA utility, and normal consumption, which lead to the CAPM (Cochrane, 2005, pp.154-155), and SDF arguments to derive the result that CARA utility together with multivariate normal payoffs works to obtain for every no-tax economy a tax economy with the same prices. I will keep the risk-free asset in zero net supply. With multivariate normal payoffs, I have to relax the assumption of a finite and discrete payoff space.

I use exponential utility of the form

$$u_i(c_i) = -\frac{\exp(-\alpha_i c_i)}{\alpha_i}, \quad (5.3.33)$$

in which $\alpha_i > 0$ is agent i 's coefficient of absolute risk aversion.

Consumption is a linear combination of multivariate normal payoffs so that consumption is

normal as well. Therefore, I rewrite the expected value in the maximization condition as

$$E[u_i(c_{i1}^*)] = -\frac{\exp(-\alpha_i E[c_{i1}^*] + 0.5\alpha_i^2 \text{Var}(c_{i1}^*))}{\alpha_i}, \quad (5.3.34)$$

with the budget constraints as in Equations (5.3.24) and (5.3.25). I maximize with respect to asset weights to obtain the first order conditions. For risky assets I obtain

$$(1 - \tau)E[\mathbf{X}_r] + \tau \mathbf{p}_r^{*\tau} - \mathbf{p}_r^{*\tau} \frac{X_0 - \tau(X_0 - p_0^{*\tau})}{p_0^{*\tau}} - \alpha_i(1 - \tau)\Omega((1 - \tau)\mathbf{n}_{ri}^* + \omega_i\tau\mathbf{n}_r) = 0, \quad (5.3.35)$$

in which Ω is the covariance matrix of the payoffs of risky assets. Using $R_f^{*\tau} = \frac{X_0 - \tau(X_0 - p_0^{*\tau})}{p_0^{*\tau}}$ and cancelling the $1 - \tau$ terms leads to

$$E[\mathbf{X}_r] - \mathbf{p}_r^{*\tau} R_f^* - \alpha_i\Omega((1 - \tau)\mathbf{n}_{ri}^* + \omega_i\tau\mathbf{n}_r) = 0, \quad (5.3.36)$$

which can be rearranged for portfolio weights

$$\mathbf{n}_{ri}^* = \frac{1}{1 - \tau} \left(\frac{1}{\alpha_i} \Omega^{-1} (E[\mathbf{X}_r] - \mathbf{p}_r^{*\tau} R_f^*) - \omega_i\tau\mathbf{n}_r \right). \quad (5.3.37)$$

The equation shows that individual portfolio weights depend on the tax rate τ , the coefficient of absolute risk aversion α_i , and the share in transfer payments ω_i . Rearranging and summing Equation (5.3.36) over all agents leads to

$$(E[\mathbf{X}_r] - \mathbf{p}_r^{*\tau} R_f^*) \sum_{i=1}^I \frac{1}{\alpha_i} = \Omega \mathbf{n}_r. \quad (5.3.38)$$

This is the same as Equation (26) in Kruschwitz and Löffler (2009) when $-\sum_{i=1}^I \frac{2}{\alpha_i} = \sum_{i=1}^I \frac{U_{iE[c]}}{U_{i\text{Var}(c)}}$ holds, in which $U_{iE[c]}$ is the derivative of a mean-variance utility function with respect to the expected value of consumption and $U_{i\text{Var}(c)}$ is the first derivative of a mean-variance utility function with respect to the variance of consumption.¹¹ Notice that Equation (5.3.34) is a mean-variance utility function. The derivatives with respect to the expected value and the variance of consumption are $U_{iE[c]} = \exp(-\alpha_i E[c_{i1}^*] + 0.5\alpha_i^2 \text{Var}(c_{i1}^*))$ and $U_{i\text{Var}(c)} = -\exp(-\alpha_i E[c_{i1}^*] + 0.5\alpha_i^2 \text{Var}(c_{i1}^*))0.5\alpha_i$. It follows that $\frac{U_{iE[c]}}{U_{i\text{Var}(c)}} = -\frac{2}{\alpha_i}$. Summing this expression over agents shows that $-\sum_{i=1}^I \frac{2}{\alpha_i} = \sum_{i=1}^I \frac{U_{iE[c]}}{U_{i\text{Var}(c)}}$ holds. It turns out that the ratio $\frac{U_{iE[c]}}{U_{i\text{Var}(c)}}$ only depends on the coefficient of absolute risk aversion α_i . Notice that Kruschwitz and Löffler (2009) state that this ratio depends on the individual agents' variances of consumption and that this is also stated in Meyer (1987) and Lajer-Chaherli and Nielsen (1993). However in those two sources the ratios presented are a bit different in that the denominator uses the derivative of the mean-variance utility function with respect to

¹¹There is a minor typo in Equations (25) and (26) in Kruschwitz and Löffler (2009). In Equation (25) the mathematical sign in front of the variance term should be positive as in Equations (13) and (16). Equation (26) has to be adjusted accordingly.

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standard deviation, i.e., $\frac{U_i E[c]}{U_i Std(c)}$, in which $Std(\cdot)$ stands for standard deviation. For the case at hand this derivative yields $U_i Std(c) = -\exp(-\alpha_i E[c_{i1}^*] + 0.5\alpha_i^2 Var(c_{i1}^*)) Std(c_{i1}^*) \alpha_i$. It follows that $\frac{U_i E[c]}{U_i Std(c)} = -\frac{1}{Std(c_{i1}^*) \alpha_i}$, which actually does depend on the standard deviation of consumption.

It turns out that for exponential utility with multivariate normal payoffs, capital gains taxes under the tax system described herein do not influence asset prices at all. As Equation (5.3.38) shows, all of the tax terms and dependencies on the tax rate disappear in the aggregate pricing equation. That leads to the following proposition:

Proposition 5.3.4. In the tax-economy set up above, in which agents have exponential utility and in which consumption only takes place at $t = 1$, the product $\mathbf{p}_r^{*\tau} R_f^*$, i.e., the ratio $\mathbf{p}_r^{*\tau}/p_0^{*\tau}$, does not depend on the tax rate. Furthermore, the corresponding no-tax economy will have the same product as the tax-economy $\mathbf{p}_r^{*\tau} R_f^* = \mathbf{p}_r R_f$, i.e., the same ratio $\mathbf{p}_r^{*\tau}/p_0^{*\tau} = \mathbf{p}_r/p_0$.

Proof. I rearrange Equation (5.3.38) to

$$\mathbf{p}_r^{*\tau} R_f^* = E[\mathbf{X}_r] - \frac{\Omega \mathbf{n}_r}{\sum_{i=1}^I \frac{1}{\alpha_i}}. \quad (5.3.39)$$

The rhs of this equation is exactly the same for the no-tax economy. The same rhs for both economies must lead to the same lhs. \square

Notice that the relation $\mathbf{p}_r^{*\tau} R_f^* = \mathbf{p}_r R_f$ just follows from the model. However, price equality is only there if the $R_f^* = R_f$. Otherwise, prices would only be proportional but not equal. Since consumption takes place only at $t = 1$ the risk-free rate does not say something about the trade-off of consumption today versus consumption tomorrow. It is exogenous to the economy and will be chosen so that prices of risk-free assets are equal.

Corollary 5.3.5. Asset prices are equal in the tax and the no-tax economy set up above, with exponential utility, multivariate normal payoffs and with consumption only at $t = 1$, when $R_f^* = R_f$, i.e., when $p_0^{*\tau} = p_0$.

Proposition 5.3.5. Given equilibrium prices, risky portfolios of all agents i for the no-tax and the tax economy are related through

$$\mathbf{n}_{ir} = \mathbf{n}_{ri}^* (1 - \tau) + \omega_i \tau \mathbf{n}_r, \quad (5.3.40)$$

and risk-free weights are related through

$$n_{0i} = n_{0i}^* (1 - \tau) + \tau (W_i^F - \omega_i W^F) \frac{1}{p_0}. \quad (5.3.41)$$

Proof. Equation (5.3.37) for a zero tax rate shows that the equation for the no-tax case is

$$\mathbf{n}_{ri} = \frac{1}{\alpha_i} \Omega^{-1} (E[\mathbf{X}_r] - \mathbf{p}_r R_f). \quad (5.3.42)$$

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Regarding the tax case, Equation (5.3.37) can be rearranged to

$$\mathbf{n}_{ri}^*(1 - \tau) + \omega_i \tau \mathbf{n}_r = \frac{1}{\alpha_i} \Omega^{-1} (E[\mathbf{X}_r] - \mathbf{p}_r^{*\tau} R_f^*). \quad (5.3.43)$$

From Proposition 5.3.4 we know that $\mathbf{p}_r^{*\tau} R_f^* = \mathbf{p}_r R_f$, and it follows that the rhs of Equation (5.3.42) and (5.3.43) are equal. Thus, the lhs of the two equations are equal as well.

For the weight on the risk-free asset, I use the budget constraint of Equation (5.3.26) but for the no-tax economy, and I substitute in Equation (5.3.15). This leads to

$$n_{0i} = \bar{n}_{0i} + (\bar{\mathbf{n}}'_{ir} - \mathbf{n}_{ri}^{*'}(1 - \tau) - \omega_i \tau \mathbf{n}'_r) \frac{\mathbf{p}_r}{p_0} \quad (5.3.44)$$

I use the relations $\bar{n}_{0i} + \bar{\mathbf{n}}'_{ir} \mathbf{p}_r / p_0 = W_i^F / p_0$ and $W^F = \mathbf{n}'_r \mathbf{p}_r$ and substitute both into the prior equation, which leads to:

$$n_{0i} = W_i^F \frac{1}{p_0} - \mathbf{n}_{ri}^{*'}(1 - \tau) \frac{\mathbf{p}_r}{p_0} - \omega_i \tau W^F \frac{1}{p_0} \quad (5.3.45)$$

Now, I add a constructive zero in the form of $n_{0i}^*(1 - \tau)p_0/p_0 - n_{0i}^*(1 - \tau)$ and rearrange to obtain the weight on the risk-free asset from the proposition

$$n_{0i} = n_{0i}^*(1 - \tau) + \tau(W_i^F - \omega_i W^F) \frac{1}{p_0}. \quad (5.3.46)$$

□

The portfolio rule for risky assets is the same as for the case with the zero risk-free rate and equal consumption and it is the same as the one presented in Kruschwitz and Löffler (2009) for the constant absolute risk aversion case. Different from Kruschwitz and Löffler (2009) I find that one only needs equal prices of the risk-free assets and price equality of risky assets follow. The weight on the risk-free assets is now different from the one presented before (Equation (5.3.16)) in that the price of the risk-free asset appears in the equation. That means individual consumptions of agents are not equal in the no-tax and the tax economy. The budget constraints are used in constructing portfolio rules and the resulting portfolio rules sum over agents to one for risky assets and to zero for the risk-free assets. It follows that budget constraints are met and markets clear.

From Equation (5.3.9) we know that $m_i^* \frac{R_f^{*\tau}}{R_f^*} = m_i$ is a sufficient condition to obtain price equality for the tax and the no-tax economy. However, the converse does not have to be true. There may be other relations of SDFs that also lead to price equality. However, under the specifications made in this section price equality also leads to $m_i^* \frac{R_f^{*\tau}}{R_f^*} = m_i$.

Proposition 5.3.6. For the type of economy set up herein, given equality of prices of a tax and a no-tax economy, the condition in Equation (5.3.9) holds.

Proof. To use the SDF language I use the budget constraint (5.3.26) in the consumption part of Equation (5.3.23) and take derivatives with respect to \mathbf{n}_{ir}^* . I obtain $\mathbf{0} = E[u'_i(c_{i1}^*)(\mathbf{X}_r^\tau - R_f^{*\tau} \mathbf{p}_r^{*\tau})]$.

I rearrange the expression to

$$\mathbf{p}_r^{*\tau} = E \left[\frac{u'_i(c_{i1}^*)}{R_f^{*\tau} E[u'_i(c_{i1}^*)]} \mathbf{X}_r^\tau \right], \quad (5.3.47)$$

in which $m_i^* = \frac{u'_i(c_{i1}^*)}{R_f^{*\tau} E[u'_i(c_{i1}^*)]}$ is the stochastic discount factor. Without taxes the SDF is $m_i = \frac{u'_i(c_{i1})}{R_f E[u'_i(c_{i1})]}$. I start with the SDFs of the tax economy and rewrite them to obtain

$$\begin{aligned} m_i^* &= \frac{u'_i(c_{i1}^*)}{R_f^{*\tau} E[u'_i(c_{i1}^*)]} \\ &= \frac{\exp(-\alpha_i c_{i1}^*)}{R_f^{*\tau} E[\exp(-\alpha_i c_{i1}^*)]} \\ &= \frac{\exp(-\alpha_i c_{i1}^*)}{R_f^{*\tau} \exp(-\alpha_i E[c_{i1}^*] + 0.5\alpha_i^2 \text{Var}(c_{i1}^*))}. \end{aligned} \quad (5.3.48)$$

The second equality uses the first derivative of the utility function (5.3.33) with respect to consumption. The third equality uses the fact that consumption is normally distributed. Consumption from Equation (5.3.25) consists of a risky part c_{i1r}^* and a risk-free part c_{i1f}^* : $c_{i1}^* = c_{i1r}^* + c_{i1f}^*$, in which $c_{i1r}^* = (\mathbf{n}_{ir}'(1-\tau) + \omega_i \tau \mathbf{n}_r') \mathbf{X}_r$ and $c_{i1f}^* = n_0^*(X_0 - \tau(X_0 - p_0^{*\tau})) + \tau \mathbf{n}_{ir}' \mathbf{p}_r^{*\tau} - \omega_i \tau \mathbf{n}_r' \mathbf{p}_r^{*\tau}$. From Equation (5.3.15) we know that the risky part of consumption in the tax and the no-tax economy are equal for any agent: $c_{i1r}^* = c_{i1r}$. That also means that $\text{Var}(c_{i1}^*) = \text{Var}(c_{i1})$. Using that I rewrite the SDF to

$$\begin{aligned} m_i^* &= \frac{\exp(-\alpha_i c_{i1r}) \exp(-\alpha_i c_{i1f}^*)}{R_f^{*\tau} \exp(-\alpha_i E[c_{i1r}]) \exp(-\alpha_i E[c_{i1f}^*]) \exp(0.5\alpha_i^2 \text{Var}(c_{i1}))} \\ &= \frac{\exp(-\alpha_i c_{i1r})}{R_f^{*\tau} \exp(-\alpha_i E[c_{i1r}]) \exp(0.5\alpha_i^2 \text{Var}(c_{i1}))}. \end{aligned} \quad (5.3.49)$$

In the second equality the $\exp(-\alpha_i E[c_{i1f}^*])$ terms cancel out. That leads to

$$\frac{R_f^{*\tau}}{R_f^*} m_i^* = \frac{\exp(-\alpha_i c_{i1r})}{R_f^* \exp(-\alpha_i E[c_{i1r}]) \exp(0.5\alpha_i^2 \text{Var}(c_{i1}))}. \quad (5.3.50)$$

The SDF of the no-tax economy can be written as

$$\begin{aligned} m_i &= \frac{u'_i(c_{i1})}{R_f E[u'_i(c_{i1})]} \\ &= \frac{\exp(-\alpha_i c_{i1})}{R_f E[\exp(-\alpha_i c_{i1})]} \\ &= \frac{\exp(-\alpha_i c_{i1})}{R_f \exp(-\alpha_i E[c_{i1}] + 0.5\alpha_i^2 \text{Var}(c_{i1}))} \\ &= \frac{\exp(-\alpha_i c_{i1r}) \exp(-\alpha_i c_{i1f})}{R_f \exp(-\alpha_i E[c_{i1r}]) \exp(-\alpha_i E[c_{i1f}]) \exp(0.5\alpha_i^2 \text{Var}(c_{i1}))} \\ &= \frac{\exp(-\alpha_i c_{i1r})}{R_f \exp(-\alpha_i E[c_{i1r}]) \exp(0.5\alpha_i^2 \text{Var}(c_{i1}))}. \end{aligned} \quad (5.3.51)$$

With price equality $R_f = R_f^*$ so that Equation (5.3.51) is equal to Equation (5.3.50). \square

5.3. Asset prices and portfolios in the no-tax and the tax economy

Some remarks on wealth are appropriate. It turns out that aggregate consumptions are the same in both economies because aggregate payoffs are the same. However, individual consumptions differ as was noted before. Consumption is also valued differently so that wealth differs between the economies. For example an agent i would value aggregate wealth as follows:

$$W^{*F} = E(m_i^* c_1) = E(m_i c_1) \frac{R_f^*}{R_f^{*\tau}} = W^F \frac{R_f^*}{R_f^{*\tau}}. \quad (5.3.52)$$

Equation (5.3.52) shows that, with a positive risk-free rate, wealth in the tax economy is greater than in the no-tax economy. Individually, equal initial portfolio holdings and price equality imply that the values of the pre-trade portfolios are the same in the tax-economy and the no-tax economy:

$$\bar{\mathbf{n}}'_i \mathbf{p}^{*\tau} = \bar{\mathbf{n}}'_i \mathbf{p}. \quad (5.3.53)$$

The increased aggregate wealth in the tax economy is due to transfer payments. I price the sum of the after-tax portfolio payoff $\bar{\mathbf{n}}'_i \mathbf{X}^\tau$ and transfer payments $Q_i = \omega_i \tau \bar{\mathbf{n}}'_i (\mathbf{X}_r - \mathbf{p}_r)$, which is an agent's total wealth, i.e., the tradeable and the nontradeable part of wealth:

$$W_i^{*F} = \bar{\mathbf{n}}'_i \mathbf{p}^{*\tau} + \omega_i \tau \bar{\mathbf{n}}'_i \left(\frac{\mathbf{p}_r^{*\tau} R_f^*}{R_f^{*\tau}} - \frac{\mathbf{p}_r^{*\tau}}{R_f^{*\tau}} \right) = W_i^{*F\tau} + W^{*F\tau} \frac{\omega_i \tau r_f^*}{R_f^{*\tau}} \quad (5.3.54)$$

$$= W_i^{*F\tau} + W^F \frac{\omega_i \tau r_f^*}{R_f^{*\tau}} \quad (5.3.55)$$

Notice that due to price equality $\bar{\mathbf{n}}'_i \mathbf{p}_r^{*\tau} = W^F = W^{*F\tau}$.

To obtain price equality agents must value the after-tax payoffs in the tax economy equally to the untaxed payoffs in the no-tax economy. For this reason the different values of total wealth must result.

5.3.3.3 Marginal utility linear in consumption

I treat the case of marginal utility linear in consumption, i.e., marginal utility of the form $u'_i(c_i) = a_i + b_i c_i$. That implies a quadratic utility function. Integration yields $u_i(c_i) = a_i c_i + 0.5 b_i c_i^2 + d_i$, in which d_i is a constant. The constant d_i just shifts the utility function up or down and has no impact on marginal utility. To have risk-averse agents the second derivative has to be negative, which leads to $u''_i(c_i) = b_i < 0$.¹² I additionally assume that consumption is nonnegative ($c_i \geq 0$) and less than or equal to bliss point consumption at $c_i \leq c_i^b = -a_i/b_i$, which is the extremum of the utility function. Since $b_i < 0$, for a positive bliss point, a_i must be positive as well. Those conditions also ensure positive marginal utility ($a_i + b_i c_i > 0$).

For an important special case of marginal utility $b_i = -1$, so that $a_i = c_i^b$ is the bliss point

¹²With risk-loving agents it is hard to ensure the existence of equilibrium because of non-convexity of preferences. Araujo et al. (2014) show examples of equilibria of economies with risk-averse and risk-loving agents.

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consumption. Agents have quadratic utility of the form

$$u_i(c_i) = -0.5(c_i - c_i^b)^2. \quad (5.3.56)$$

This leads to marginal utility linear in consumption of the form

$$u'_i(c_i) = c_i^b - c_i. \quad (5.3.57)$$

I use this specification for the tax economy in Equation (5.3.27) and simplify to obtain

$$E[(c_i^b - c_i^*)(\mathbf{X}_r - \mathbf{p}_r^{*\tau}/p_0^{*\tau})] = 0. \quad (5.3.58)$$

Summing over agents leads to

$$E[(c^b - c^*)(\mathbf{X}_r - \mathbf{p}_r^{*\tau}/p_0^{*\tau})] = 0. \quad (5.3.59)$$

This leads to the following proposition:

Proposition 5.3.7. In the tax-economy set up above, in which agents have quadratic utility and in which consumption only takes place at $t = 1$, the product $\mathbf{p}_r^{*\tau} R_f^*$, i.e., the ratio $\mathbf{p}_r^{*\tau}/p_0^{*\tau}$, does not depend on the tax rate. Furthermore, the corresponding no-tax economy will have the same product as the tax-economy $\mathbf{p}_r^{*\tau} R_f^* = \mathbf{p}_r R_f$, i.e., the same ratio $\mathbf{p}_r^{*\tau}/p_0^{*\tau} = \mathbf{p}_r/p_0$.

Proof. Since aggregate consumptions are the same in the tax and the no-tax economy, i.e., $c^* = c$, the term $\mathbf{p}_r^{*\tau}/p_0^{*\tau}$ in Equation (5.3.59) also does not depend on a tax rate. \square

Quadratic utility implies a mean-variance utility function and the CAPM. It was not mentioned in Kruschwitz and Löffler (2009) as producing equilibria with equal prices. However, the term $\sum_{i=1}^I \frac{U_{iE[c]}}{U_{iVar(c)}}$ is also independent from the tax rate for quadratic utility. Taking expectations of Equation (5.3.56) and expanding the quadratic expression leads to

$$E[u_i(c_i)] = -0.5E[c_i^2] + c_i^b E[c_i] - 0.5c_i^{b2} \quad (5.3.60)$$

$$= -0.5Var(c_i) - 0.5E[c_i]^2 + c_i^b E[c_i] - 0.5c_i^{b2}. \quad (5.3.61)$$

It follows that $\frac{U_{iE[c]}}{U_{iVar(c)}} = 2(E[c_i] - c_i^b)$ and $\sum_{i=1}^I \frac{U_{iE[c]}}{U_{iVar(c)}} = 2(E[c] - c^b)$. Aggregating c_i^* this way would lead to aggregate consumption in the tax economy, i.e., to c^* . But aggregate consumptions are the same so that $\sum_{i=1}^I \frac{U_{iE[c]}}{U_{iVar(c)}} = \sum_{i=1}^I \frac{U_{iE[c^*]}}{U_{iVar(c^*)}}$ or $2(E[c] - c^b) = 2(E[c^*] - c^b)$.

This can be generalized to any function with marginal utility linear in consumption:

Proposition 5.3.8. In the tax-economy set up above, in which agents have utility that leads to linear marginal utility and in which consumption only takes place at $t = 1$, the product $\mathbf{p}_r^{*\tau} R_f^*$, i.e., the ratio $\mathbf{p}_r^{*\tau}/p_0^{*\tau}$, does not depend on the tax rate. Furthermore, the corresponding no-tax

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economy will have the same product as the tax-economy $\mathbf{p}_r^{*\tau} R_f^* = \mathbf{p}_r R_f$, i.e., the same ratio $\mathbf{p}_r^{*\tau}/p_0^{*\tau} = \mathbf{p}_r/p_0$.

Proof. A single agent's pricing equation is

$$\mathbf{p}_r^{*\tau}/p_0^{*\tau} = \frac{E[(a_i + b_i c_i^*) \mathbf{X}_r]}{a_i + b_i E[c_i^*]}. \quad (5.3.62)$$

I aggregate in a fashion similar to Appendix 5.7

$$\mathbf{p}_r^{*\tau}/p_0^{*\tau} = E \left[\mathbf{X}_r \frac{\sum_{i=1}^I \frac{a_i}{b_i} + c}{\sum_{i=1}^I \frac{a_i}{b_i} + E[c]} \right], \quad (5.3.63)$$

The remainder is the same as in the proof of Proposition 5.3.7. □

For the portfolio rule it suffices to look at consumption, since marginal utility is a simple linear function of consumption. I use the budget constraint (5.3.26) in Equation (5.3.25), which leads to

$$c_i^* = \bar{n}_{i0} X_0^\tau + \left(\frac{1}{p_0^{*\tau}} (1 - \tau) (\bar{\mathbf{n}}'_{ir} - \mathbf{n}'_{ir}) + \tau \bar{\mathbf{n}}'_{ir} \right) \mathbf{p}_r^{*\tau} + (\mathbf{n}'_{ir} (1 - \tau) + \omega_i \tau \mathbf{n}'_r) \mathbf{X}_r - \omega_i \tau \mathbf{n}'_r \mathbf{p}_r^{*\tau} \quad (5.3.64)$$

The term $\mathbf{n}'_{ir} (1 - \tau) + \omega_i \tau \mathbf{n}'_r$ attached to \mathbf{X}_r shows that the portfolio rule for risky asset shares is the same as in Proposition 5.3.5. Substituting that into the budget constraint also leads to one for the risk-free asset as in Proposition 5.3.5. Thus, budget constraints are considered and markets clear as before.

5.3.4 Other utility functions

The cases presented before led to aggregate pricing functions that were independent from the tax rate. Therefore, together with the convenience that the price of the risk-free asset can be chosen, asset prices are the same in the tax and the no-tax economy. For other utility functions, which lead to nonlinear marginal utility, aggregation does not generally lead to an aggregated SDF that is linear in aggregate consumption. For example for power utility of the form $u_i(c_i) = (c_i)^{1-\alpha_i}/(1-\alpha_i)$ marginal utility is $c_i^{-\alpha_i}$. Aggregation leads to $\sum_{i=1}^I c_i^{-\alpha_i}$, which does not generally allow a statement in terms of aggregate consumption. Thus, if I use consumption in the tax economy c_i^* , the tax terms remain in the pricing equation, and prices remain dependent on taxes.

5.4 Discussion of the results and limitations of the analyses

5.4.1 Some general remarks

I begin with a general discussion of the results especially with regard to the results derived in Kruschwitz and Löffler (2009). I show in particular that the results in Kruschwitz and Löffler

(2009), which are derived for mean-variance optimizing agents, also hold for agents maximizing expected utility over consumption. I additionally stress the result, that a zero risk-free rate leads to tax payments with a price of zero. That makes pre- and after-tax prices as well as wealth before and after taxes and transfers equal. For expositional reasons, I chose to compare an economy with a tax rate of zero and one with a non-zero tax rate. However, the argumentation above can be carried out for any pair of different tax rates. There is no need that one has to be zero. That should be already clear from the fact that to obtain price equality in the tax and the no-tax economy it does not matter what the value of τ is.

A zero risk-free rate leads to zero prices of capital gains taxes.¹³ By the same token, one could argue that a risk-free rate going to zero makes prices of tax payments going to zero as well. This suggests that price distortions through the tax can be reduced and even be eliminated through changing the risk-free rate towards zero. However, there are some problems with this argumentation. First, when there is the possibility of more than one equilibrium, one cannot be sure that the introduction of capital gains taxes or a change of the tax rate will lead to the equilibrium that corresponds to the equilibrium before the change and therefore leaves prices unchanged. Kruschwitz and Löffler (2009) also mention that the CAPM typically has several equilibria with different price vectors. They show that there exists at least one equilibrium with an unchanged price vector when taxes are introduced. The CAPM does not rely on complete markets so that there is no market completeness that would contribute to a unique solution. For the case herein, if there is a single solution, the portfolio changes derived herein are consistent with price equality and with the same consumption streams of agents before and after the tax rate change. The second problem is that the argumentation of making the risk-free rate zero leaves the question open on how this is done. If the risk-free rate is exogenous to the economy, one can leave all else equal and change the rate towards zero to reduce price distortions. However, if the risk-free rate is endogenous one cannot just change the risk-free rate leaving all else equal. Then, there exist some more fundamental causes that change the risk-free rate.

In practice, the tax system is much more complex than the one set up here. For example, the tax system outlined here is symmetric in the way that positive and negative capital gains are taxed equally. This is crucial to obtain the result of a zero price of taxes with a zero risk-free rate. In the U.S. negative capital gains can be deducted from taxes only up to 3000 USD and there need to be taxes paid in the first place to have something to deduct the losses from.¹⁴ Introducing more periods and tax losses that can be carried forward complicates the analysis even more. Thus, a more realistic tax system at least weakens the conclusions drawn here. The consequences of a more realistic tax code show scope for future research on the topic.

¹³A zero risk-free rate is difficult to obtain. Even though rates are currently very low especially in Europe, this is not the usual condition. There are many further issues such as that we only observe proxies of risk-free rates and that those rates usually have a non-flat term structure.

¹⁴For further reference on rules for the treatment of capital gains and losses in the U.S. see <https://www.irs.gov/taxtopics/tc409.html>.

5.4. Discussion of the results and limitations of the analyses

For the model with consumption at $t = 0$ and $t = 1$ the risk-free rate is endogenous. I show that with linear marginal utility, there can never be price equality for a risk-free rate not equal to zero. This is because the prices of the risk-free assets differ in the tax and the no-tax economy. However, when consumption takes place only at $t = 1$ so that the risk-free assets' prices are exogenous and chosen to be equal, price equality can be established for linear marginal utility as well.

Without a zero risk-free rate but with exponential utility (CARA utility) and multivariate normal payoffs there is again an equilibrium that is consistent with unchanged prices. Exponential utility implies constant absolute risk aversion and increasing relative risk aversion (IRRA) (Lengwiler, 2004, p.92). This utility specification is convenient because together with the normality assumption it leads to the CAPM. But it is not how we think about the characteristics of economic agents. Decreasing absolute risk aversion (DARA) and constant relative risk aversion (CRRA) seem intuitively and empirically to be more realistic (Lengwiler, 2004, p.87).

Having noted this, and looking at the current situation with relatively low real and nominal rates in Europe and the U.S., I will provide a simple numerical example on asset price distortions through different capital gains tax rates and different risk-free rates.

5.4.2 A simple example with quadratic utility and consumption at $t = 0$ and $t = 1$

5.4.2.1 Model specification and base case with a zero risk-free rate

I use a simple model with two agents $I = 2$, two risky assets $N = 2$, and a risk-free asset. There are three states and the payoffs are not redundant so that the asset market is complete. I set the time discount factors of the agents to one, so that I do not have to consider them explicitly here. I summarize the characteristics of the two agents in Table 5.1.

Table 5.1: Characteristics of agents

| Parameter | Agent 1 | Agent 2 | Aggregate |
|-----------------------|---------|---------|-----------|
| c_i^b | 1.5 | 2 | 3.5 |
| \bar{c}_{i0} | 0 | 1 | 1 |
| $\bar{\mathbf{n}}_i'$ | (0 1 1) | (0 0 0) | (0 1 1) |

The payoff space is specified as follows:

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 \\ 0.4 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \quad (5.4.1)$$

As before rows are assets and columns are states. Since the risk-free asset is in zero net supply, aggregate consumption in the three states is $c_1 = (0.4 \ 2 \ 3)$. The linear marginal utility of the

5.4. Discussion of the results and limitations of the analyses

agents leads to simple expressions for the SDFs with

$$m_i = \frac{c_i^b - c_{i1}}{c_i^b - c_{i0}}, \quad (5.4.2)$$

for $i = 1, 2$. Due to market completeness the SDFs of the two agents are equal. Furthermore, due to the linearity of marginal utility of consumption, the SDF can be represented through aggregate consumption and aggregate bliss point consumption:

$$m_i = \frac{c_i^b - c_{i1}}{c_i^b - c_{i0}} = \frac{c^b - c_1}{c^b - c_0}, \quad (5.4.3)$$

for $i = 1, 2$. To see that one can write down the vector of prices in the form

$$\mathbf{p} = E \left[\begin{array}{c} c_i^b - c_{i1} \\ c_i^b - c_{i0} \end{array} \mathbf{X} \right] \quad (5.4.4)$$

and multiply both sides by $c_i^b - c_{i0}$. The resulting expression is $(c_i^b - c_{i0})\mathbf{p} = E [(c_i^b - c_{i1})\mathbf{X}]$ for $i = 1, 2$. Now, one can simply add up those expressions for the two agents to obtain $(c_1^b - c_{10} + c_2^b - c_{20})\mathbf{p} = E [(c_1^b - c_{11} + c_2^b - c_{21})\mathbf{X}]$, which is the same as $(c^b - c_0)\mathbf{p} = E [(c^b - c_1)\mathbf{X}]$, and can be rearranged to

$$\mathbf{p} = E \left[\frac{c^b - c_1}{c^b - c_0} \mathbf{X} \right]. \quad (5.4.5)$$

I summarize statewise the probabilities π_s , the elements of the SDF m_s and state prices κ_s for $s = 1, 2, 3$ in Table 5.2. The SDF is given through aggregate consumption and the aggregate bliss points. I choose the probabilities so that I obtain a zero risk-free rate, i.e., a gross risk-free rate of one. This can be verified through summing up the state prices in the table, which sum up to one. That allows to compute the asset prices, which are $\mathbf{p}' = (1.00 \ 0.63 \ 0.05)$. To

Table 5.2: Summary of pricing components

| States s | π_s | m_s | κ_s |
|------------|---------|-------|------------|
| 1 | 0.67 | 1.24 | 0.83 |
| 2 | 0.25 | 0.60 | 0.15 |
| 3 | 0.08 | 0.20 | 0.02 |

obtain portfolio weights I use the fact that $m = (c_i^b - c_{i1})/(c_i^b - c_{i0})$. I substitute $c_{i1} = \mathbf{n}'_i \mathbf{X}$ and $c_{i0} = (\bar{\mathbf{n}}'_i \mathbf{p} + \bar{c}_{i0}) - \mathbf{n}'_i \mathbf{p}$ into this expression and rearrange for portfolio weights. This yields

$$\mathbf{n}'_i = (c_i^b + ((\bar{\mathbf{n}}'_i \mathbf{p} + \bar{c}_{i0}) - c_i^b)m)(\mathbf{p}m + \mathbf{X})^{-1}. \quad (5.4.6)$$

I use this equation to compute portfolio weights \mathbf{n}_i , initial consumption c_{i0} , wealth after initial consumption W_i^F , and consumption at $t = 1$ for the different states c_{i1} . Table 5.3 summarizes this information. The table shows that the agents put equal weights on the risky assets. Quadratic

5.4. Discussion of the results and limitations of the analyses

Table 5.3: Summary of agents' portfolios, consumption and wealth in the no-tax economy

| Agent i | \mathbf{n}_i | c_{i0} | W_i^F | c_{i1} |
|-----------|-------------------|----------|---------|------------------|
| 1 | (-0.03 0.44 0.44) | 0.41 | 0.27 | (0.15 0.85 1.28) |
| 2 | (0.03 0.56 0.56) | 0.59 | 0.41 | (0.25 1.15 1.72) |
| 1+2 | (0.00 1.00 1.00) | 1.00 | 0.68 | (0.40 2.00 3.00) |

utility implies the CAPM (Cochrane, 2005, p. 153), which, in turn, implies that every agent holds a share of the portfolio of risky assets. Thus, every agent must hold the same share of every risky asset.

In the tax economy the SDF does not change because aggregate consumption does not change. I again use the fact that $m^* = m = (c_i^b - c_{i1}^*) / (c_i^b - c_{i0}^*)$. The tax rate τ is 40%. I substitute $c_{i1}^* = \mathbf{n}_i^{*\prime}(\mathbf{X} - \tau(\mathbf{X} - \mathbf{p}^{*\tau})) + \omega_i \tau \mathbf{n}_i'(\mathbf{X} - \mathbf{p}^{*\tau})$ and $c_{i0} = (\bar{\mathbf{n}}_i' \mathbf{p}^{*\tau} + \bar{c}_{i0}) - \mathbf{n}_i' \mathbf{p}^{*\tau}$ into this expression and rearrange for portfolio weights. This yields

$$\mathbf{n}_i^{*\prime} = (c_i^b + ((\bar{\mathbf{n}}_i' \mathbf{p}^{*\tau} + \bar{c}_{i0}) - c_i^b) m - \omega_i \tau \mathbf{n}_i'(\mathbf{X} - \mathbf{p}^{*\tau})) (\mathbf{p}^{*\tau} m + \mathbf{X}(1 - \tau) + \tau \mathbf{p}^{*\tau})^{-1}. \quad (5.4.7)$$

I use this expression to compute portfolio weights \mathbf{n}_i^* , initial consumption c_{i0}^* , wealth after consumption $W_i^{*F\tau}$, and consumption at $t = 1$ for the different states. The results are presented in Table 5.4. Expectedly, the only change compared to the result in Table 5.2 is the change in the

Table 5.4: Summary of agents' portfolios, consumption and wealth in the tax economy

| Agent i | \mathbf{n}_i^* | c_{i0}^* | W_i^{*F} | c_{i1}^* |
|-----------|-------------------|------------|------------|------------------|
| 1 | (-0.09 0.53 0.53) | 0.41 | 0.27 | (0.15 0.85 1.28) |
| 2 | (0.09 0.47 0.47) | 0.59 | 0.41 | (0.25 1.15 1.72) |
| 1+2 | (0.00 1.00 1.00) | 1.00 | 0.40 | (0.40 2.00 3.00) |

portfolio weights. Alternatively to Equation (5.4.7), one can also use the expression in Proposition 5.3.3 rearranged for $\mathbf{n}_i^{*\prime}$, which yields exactly the same portfolios weights, which, in turn, imply equal consumption in both economies. I continue to look at effects of changes in the risk-free rate on prices.

5.4.2.2 Price distortions through a non-zero risk-free rate

In the constructed model the risk-free rate is endogenous. Thus, there must be a change in the basic economy to induce a change in the risk-free rate. I alter the risk-free rate through using different probabilities of the states, and I hold the remaining parameters constant. This leaves the SDF unchanged but leads to new state prices. Notice that with quadratic utility and $\beta_i = 1$ for all i , $E[m] = 1$ means that

$$E \left[\frac{c^b - c_1}{c^b - c_0} \right] = 1, \quad (5.4.8)$$

which implies that $E[c_1] = c_0$ or zero expected consumption growth. Changing bliss point consumption changes prices but will eventually not lead to a zero risk-free rate. With an impatience factor β equal for all agents and not equal to one, it is possible to change bliss point consumption so that a zero risk-free rate can be obtained.

The endogeneity of the risk-free rate makes clear that a change in the risk-free rate is triggered by some fundamental change in the economy. Thus, it is not instructive to compare the pair of a no-tax and a tax economy at a zero risk-free rate with a pair of economies at a different risk-free rate. Furthermore, the same value for the risk-free rate can be obtained through different changes of fundamental parameters, leading to different prices and price differences in both economies. Therefore, I restrict the analysis to changes in the probabilities of states one and three.

To compute a target risk-free rate R_f^T , I use the fact that $1/R_f^T = E[m] = m_1\pi_1 + m_2\pi_2 + m_3\pi_3$, where the subscripts in the last equality are states. Probabilities must sum to one so that $1/R_f^T = m_1\pi_1 + m_2\pi_2 + m_3(1 - \pi_1 - \pi_2)$. I rearrange for π_1 :

$$\pi_1 = \frac{1/R_f^T - m_2\pi_2 - m_3 + m_3\pi_2}{m_1 - m_3}. \quad (5.4.9)$$

Probability π_2 remains unchanged and $\pi_3 = 1 - \pi_1 - \pi_2$.

The model with quadratic utility is very convenient to compute price differences because the discount factor does not change when going from the no-tax to the tax economy. Since the SDF does not change, the pre-tax price in the tax economy is equal to the price in the no-tax economy: $\mathbf{p}^* = \mathbf{p}$. That holds for any risk-free rate, but it is special to the quadratic utility. Therefore, I can use Equation (5.2.17), in which I substitute in p_j for p_j^* and $1/R_f^{*\tau}$ for $E[m^*]$:

$$p_j^{*\tau} = \frac{p_j(1 - \tau)}{1 - \tau/R_f^{*\tau}}. \quad (5.4.10)$$

I rearrange Equation (5.4.10) to an expression that yields the price difference in percent of price in the tax economy:

$$\frac{p_j - p_j^{*\tau}}{p_j^{*\tau}} = \frac{1 - \tau/R_f^{*\tau}}{1 - \tau} - 1. \quad (5.4.11)$$

Thus, for quadratic utility the percentage change in price versus the no-tax case is the same for any asset j . As the prior equation shows, the rhs does not depend on any asset specific parameter.

The unchanged SDF between the no-tax and the tax economy also means that the risk-free rate in the no-tax economy is equal to the after-tax risk-free rate in the tax economy: $R_f = 1/E[m] = 1/E[m^*] = R_f^{*\tau}$. I show the results for some ranges of the after-tax risk-free rate and the tax rate in Table 5.5. I also provide the implied pre-tax risk-free rates on the rhs of the table.

The table shows that a higher risk-free rate increases the price difference between prices in the tax economy and those in the no-tax economy. For example for $r_f^{*\tau} = 20\%$ and a tax rate of 5% the prices in the no-tax economy are 0.9% higher than in the tax economy. For $r_f^{*\tau} = -5\%$ the

5.4. Discussion of the results and limitations of the analyses

Table 5.5: Price changes through capital gains taxes for different risk-free rates

| | Price changes $\frac{p_j - p_j^{*\tau}}{p_j^{*\tau}}$ | | | | | | Implied pre-tax risk-free rate r_f^* | | | | | |
|------------|---|-----|-----|-----|-----|------|--|-----|-----|------|------|------|
| | $r_f^{*\tau} =$ | | | | | | $r_f^{*\tau} =$ | | | | | |
| | -5 | 0 | 5 | 10 | 15 | 20 | -5 | 0 | 5 | 10 | 15 | 20 |
| $\tau =$ 5 | -0.3 | 0.0 | 0.3 | 0.5 | 0.7 | 0.9 | -5.3 | 0.0 | 5.3 | 10.5 | 15.8 | 21.1 |
| 10 | 0.6 | 0.0 | 0.5 | 1.0 | 1.4 | 1.9 | -5.6 | 0.0 | 5.6 | 11.1 | 16.7 | 22.2 |
| 20 | -1.3 | 0.0 | 1.2 | 2.3 | 3.3 | 4.2 | -6.3 | 0.0 | 6.3 | 12.5 | 18.8 | 25.0 |
| 30 | -2.3 | 0.0 | 2.0 | 3.9 | 5.6 | 7.1 | -7.1 | 0.0 | 7.1 | 14.3 | 21.4 | 28.6 |
| 40 | -3.5 | 0.0 | 3.2 | 6.1 | 8.7 | 11.1 | -8.3 | 0.0 | 8.3 | 16.7 | 25.0 | 33.3 |

All numbers are in percent and rounded to one decimal.

prices in the no-tax economy are 0.3% less than the ones in the tax economy. The columns of the table show that the magnitudes increase with an increasing tax rate.

I show the effects of the combination $\tau = 40\%$ and $r_f^{*\tau} = 20\%$ a bit more in detail. The price vector in the no-tax economy is then $\mathbf{p}' = (0.83 \ 0.55 \ 0.14)$, which means that assets are worth 0.69 (the last two prices). In the tax economy the price vector is $\mathbf{p}^{*\tau} = (0.75 \ 0.50 \ 0.13)$, so that the total value of financial assets amounts to 0.63. The differences of the values reflect the 11.1% given in Table 5.5.

Table 5.6: Summary of agents' portfolios, consumption and wealth in the no-tax and the tax economy

| No-tax economy | | | | |
|----------------|-------------------|------------|------------|------------------|
| Agent i | \mathbf{n}_i | c_{i0} | W_i^F | c_{i1} |
| 1 | (-0.02 0.44 0.44) | 0.41 | 0.28 | (0.15 0.85 1.28) |
| 2 | (0.02 0.56 0.56) | 0.59 | 0.41 | (0.25 1.15 1.72) |
| Tax economy | | | | |
| Agent i | \mathbf{n}_i^* | c_{i0}^* | W_i^{*F} | c_{i1}^* |
| 1 | (-0.13 0.54 0.54) | 0.39 | 0.24 | (0.12 0.83 1.28) |
| 2 | (0.13 0.46 0.46) | 0.61 | 0.39 | (0.28 1.17 1.72) |

The assumed tax rate is 40% and $r_f^{*\tau} = 20\%$.

Consumption in $t = 0$ and $t = 1$ as well as wealth are different between the economies. Notice, however, that the agents' consumption does not change a lot (Table 5.6). They try to stick to the pattern that was optimal without taxes. The tax transfers, which now have a non-zero price, cannot be traded at $t = 0$ so that agents have to rearrange what they want to consume now and what they want to save for the next period. With quadratic utility they can do that without changing the SDF. In the no-tax economy agents can trade all of the claims on future consumption, which leads to a different division of endowments into initial consumption and wealth, as well as to a changed future consumption.

Still with a complete asset market but with a more demanding utility function such as power utility, aggregation of marginal utilities is more complicated and does not involve just summing up

consumption of agents. It involves summing over nonlinear functions of consumption. That means introducing taxes will likely not leave the SDF unchanged, which in turn can lead to different risk-free rates in the two economies. Then, something like Table 5.5 needs a new interpretation because $r_f^{*\tau} = r_f$ is not likely to hold anymore.

5.5 Conclusion

I extend the analysis of Kruschwitz and Löffler (2009) about the effects of capital gains taxes on asset pricing. Specifically, they analyze cases when taxes do not change prices in a mean-variance CAPM with consumption only at one future point in time. I extend the analysis to agents who maximize expected utility over consumption. I construct two economies that have equal endowments, utility functions of investors and payoffs. I look at economies with consumption at $t = 0$ and $t = 1$. I find that, with a risk-free rate of zero, taxes on returns have zero value. Using this result, I can show that for a no-tax economy there exists a tax economy with the same asset prices. This is the case when investors change their portfolios to consume the same in the tax economy in every state as in the no-tax economy. This implies the same portfolio rule as proposed in Kruschwitz and Löffler (2009). With a non-zero risk-free rate equilibria with equal prices and in which agents consume the same in the tax as in the no-tax economy do not generally exist. In the case of linear marginal utility, prices are always different in both economies for a non-zero risk-free rate.

Furthermore, I show that with exponential utility, multivariate normal payoffs, and consumption only in $t = 1$, prices are equal when the risk-free rate in both economies are the same. But this time taxes do not have zero value so that aggregate wealth after initial consumption is different in both economies. Individual consumption profiles as well as SDFs are different in the two economies. I find that the portfolio rule for risky assets is the same as in Kruschwitz and Löffler (2009) and the same is true for the case with a zero risk-free rate and in which all agents consume the same in both economies in every state. Additionally, utility functions that lead to linear marginal utility also have equilibria with equal prices. Aggregation makes the pricing equation independent from the tax rate. With nonlinear marginal utility, price equality cannot generally be obtained since aggregation will keep the tax rate terms in the aggregate pricing equation. Eventually, also in the case with consumption only at $t = 1$, a zero risk-free rate leads equal price vectors for the two economies.

5.6 Appendix - Alternative derivation for the exponential utility case with multivariate normal payoffs

Another approach to obtain the same the pricing equation but to keep more track of the SDF language is to use Stein's lemma. The SDF is a function of consumption, which, in turn, is normally distributed. The vector \mathbf{X}_r^τ still contains the tax expression so that I can rearrange $\mathbf{p}_r^{*\tau} = E[m_i^* \mathbf{X}_r^\tau]$ to obtain the known equation $\mathbf{p}_r^{*\tau} = E\left[m_i^* \mathbf{X}_r \frac{R_f^{*\tau}}{R_f^*}\right]$.

I use the definition for the SDF, i.e., $m_i^* = \frac{u'_i(c_i^*)}{R_f^{*\tau} E[u'_i(c_i^*)]}$. Assuming multivariate normal payoffs, I use Stein's Lemma for the pricing equation:

$$\begin{aligned} \mathbf{p}_r^{*\tau} &= E\left[m_i^* \mathbf{X}_r \frac{R_f^{*\tau}}{R_f^*}\right] = \frac{E[\mathbf{X}_r]}{R_f^*} + Cov(m_i^*, \mathbf{X}) \frac{R_f^{*\tau}}{R_f^*} \\ &= \frac{E[\mathbf{X}_r]}{R_f^*} + Cov(u'_i(c_i^*), \mathbf{X}_r) \frac{1}{R_f^* E[u'_i(c_i^*)]} \\ &= \frac{E[\mathbf{X}_r]}{R_f^*} + E[u'_i(c_i^*)] Cov(c_{i1}^*, \mathbf{X}_r) \frac{1}{R_f^* E[u'_i(c_i^*)]} \\ &= \frac{E[\mathbf{X}_r]}{R_f^*} + E[u'_i(c_i^*)] \Omega(\mathbf{n}_{ri}^*(1 - \tau) + \omega_i \tau \mathbf{n}_r) \frac{1}{R_f^* E[u'_i(c_i^*)]}. \end{aligned} \quad (5.6.1)$$

Marginal utility of consumption at $t = 1$ is

$$u'_i(c_{i1}^*) = \exp(-\alpha_i c_{i1}^*). \quad (5.6.2)$$

The second derivative with respect to c_{i1}^* is

$$u''_i(c_{i1}^*) = -\alpha_i \exp(-\alpha_i c_{i1}^*) = -\alpha_i u'_i(c_{i1}^*). \quad (5.6.3)$$

I substitute this back into Equation (5.6.1) to obtain

$$\mathbf{p}_r^{*\tau} = \frac{E[\mathbf{X}_r]}{R_f^*} - \frac{\alpha_i}{R_f^*} \Omega(\mathbf{n}_{ri}^*(1 - \tau) + \omega_i \tau \mathbf{n}_r). \quad (5.6.4)$$

This is the same as Equation (5.3.36), which can be aggregated to Equation (5.3.38). For the no-tax economy using Stein's Lemma gives

$$\mathbf{p} = \frac{E[\mathbf{X}]}{R_f} + -\frac{\alpha_i}{R_f} \Omega \mathbf{n}_{ri}. \quad (5.6.5)$$

With equal prices so that also $R_f^* = R_f$, Equations (5.6.1) and (5.6.5) lead to

$$\mathbf{n}_{ri}^*(1 - \tau) + \omega_i \tau \mathbf{n}_r = \mathbf{n}_{ri}, \quad (5.6.6)$$

which is expectedly the same portfolio rule derived earlier. Using the budget constraints leads to the same rule for the risk-free asset as before.

5.7 Appendix - Linear marginal utility and aggregation

I denote a general form of linear marginal utility as $u'(c_{i1}) = a_i + b_i c_{i1}$, so that the individual SDF for the model with consumption at times $t = 0$ and $t = 1$ is

$$m_{i1} = \beta \frac{a_i + b_i c_{i1}}{a_i + b_i c_{i0}}. \quad (5.7.1)$$

Putting that into Equation (5.2.5) and multiplying by $(a_i + b_i c_{i0})$, I obtain

$$\mathbf{p}(a_i + b_i c_{i0}) = E[\mathbf{X}\beta(a_i + b_i c_{i1})]. \quad (5.7.2)$$

I divide by b_i and and sum over all i to obtain

$$\mathbf{p}\left(\sum_{i=1}^I \frac{a_i}{b_i} + c_0\right) = E\left[\mathbf{X}\beta\left(\sum_{i=1}^I \frac{a_i}{b_i} + c_1\right)\right], \quad (5.7.3)$$

in which $c_0 = \sum_{i=1}^I c_{i0}$ and $c_1 = \sum_{i=1}^I c_{i1}$. Rearrangig again leads to

$$\mathbf{p} = E\left[\mathbf{X}\beta \frac{\sum_{i=1}^I \frac{a_i}{b_i} + c_1}{\sum_{i=1}^I \frac{a_i}{b_i} + c_0}\right]. \quad (5.7.4)$$

Aggregate values for consumption can be used in the SDF. For the form of quadratic utility that is used in Cochrane (2014) and also herein, i.e., for $u(c_{it}) = -0.5(c_{it} - c_i^b)^2$, marginal utility is $c_{it} - c_i^b$, so that $b_i = -1$ for all i and $a_i = c_i^b$. Notice that the time discount factors β are equal for all agents. With different time discount factors the equation changes to

$$\mathbf{p} = E\left[\mathbf{X} \frac{\sum_{i=1}^I \frac{a_i}{b_i} + c_1}{\sum_{i=1}^I \frac{1}{\beta_i} \left(\frac{a_i}{b_i} + c_{0i}\right)}\right], \quad (5.7.5)$$

which shows that aggregate consumption at $t = 0$ cannot generally be used in the SDF.

Chapter 6

Conclusion

6.1 Summary

I will give a quick summary here and then focus on possible future research in the next section. I provide four articles that analyze how taxes affect asset prices. The first article investigates the applicable discount rate to discount (expected) tax savings, when a company has risky debt. The article makes clear that the usual discount rates such as the risk-free rate or the cost of debt are only correct in very special cases. Without taxing cancelled debt, tax savings are proportional to interest payments and the discount rate for tax savings has to be the same as for interest payments. Priority of interest over principal payments or vice versa strongly affects the discount rate on tax savings. Only with a proportional loss distribution on principal and interest payments are tax savings also to be discounted at the discount rate for debt. The second article investigates the pricing of tax savings and further focusses on the effects of taxes on cancelled debt. Taxing cancelled debt decreases tax savings, but it also makes pricing more convenient. The resulting pricing equation for tax savings is independent from the distribution of losses on interest and principal payments. Without the tax on cancelled debt the loss distribution matters. A convenient equation with the coupon rate as a parameter can be obtained for pro rata losses. For other distributions, additional assumptions are needed to simplify the pricing equation or one has to use more elaborate pricing techniques. The third article uses the results from the first two articles to extend the de- and re-levering procedures that are already around to more cases. I derive the procedure for situations when cancelled debt is not taxed and losses are distributed proportionally, through prioritizing interest over principal payments, and through prioritizing principal over interest payments. In the last article I look at an endowment economy as a whole. I investigate situations in which the introduction of a capital gains tax may lead to an equilibrium with the same price vector as before the introduction of taxes. I do this in a framework of agents maximizing expected utility over consumption. As was found before for the mean-variance CAPM, a zero risk-free rate leads to an unchanged price vector as well as CARA utility in the form of exponential utility with multivariate normal payoffs. Additionally, quadratic utility also leads to an unchanged price vector.

6.2. Potential future research

The first three articles illuminate especially the corporate finance literature. They show, for example, that the usual WACC textbook equations are derived under a special set of assumptions on taxes that may or may not be applicable to the economy one is looking at. Thus, there is much scope for practical application here. The last article shades light on more theoretic questions. The results may be more important for a modeler to know the behavior of the usual economic models under certain specifications. Having stated this, I will turn to the potential for future research in the field of asset pricing and tax effects.

6.2 Potential future research

The first three articles are mainly, but with some important extensions, based on assumptions that I would call the usual corporate finance textbook assumptions. They include the independence of investment and financing decisions, constant discount rates and constant tax rates over time, as well as the exclusion of personal tax rates. The constant discount rate assumption could further be justified by additional assumptions such as independently and identically distributed returns on the relevant assets. All of this helps practitioners to reduce complexity and to perform valuations. Even though a model is per definition a simplification of reality, it should consider all of the factors that have a substantial impact on the value of an asset. Including more of those important factors may be a fruitful field for future research. I will mention several of those factors, and I will detail the ones that are close to the topics of the respective articles.

From a more macro perspective, one could abandon the i.i.d. world assumption and condition expected returns on business cycle stages and capital market expectations. That also means to leave the assumption of a constant risk-free rate, i.e., a flat yield curve in favor of a sloped yield curve that expresses market expectations on future yields. In an i.i.d. world, which has a flat yield curve, an investor is indifferent in holding long- or short-term debt. They have both the same risk and expected return characteristics. When this assumption is relaxed decisions between long- and short-term debt become relevant.

Furthermore, tax rates are not constant over time. Instead, tax legislation changes with some frequency that one may measure in years. Expectations can also be formed on the applicable tax rate and its behavior over business cycles. Such models could be based on Sialm (2006), who develops a simple dynamic general equilibrium model based on log-normal growth of output. He finds that stochastic tax rates may increase the equity premium and may partly explain the equity premium puzzle. More on the modelling of stochastic tax rates can be found in Niemann (2004) and Niemann (2011). Krause (2017) also investigates asset pricing under tax rate uncertainty. In a first model, tax rate cyclicalities are analyzed. When taxes are transferred back to the investor countercyclical tax rates regularly lower discount rates and increase asset prices. However, those patterns do not hold anymore when taxes are not transferred back. Furceri and Karrast (2011) actually test three theories on fiscal policy and the effective average tax rate along the business cycle. They find that the correlation of tax rates with cyclical output is relatively small and

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statistically insignificant. Furthermore, they find that to be in line with the hypothesis that propagates tax smoothing. Sialm (2009) also carries out an empirical study of the effect of tax changes on asset prices. He only looks at taxes that are closely related to asset prices such as taxes on dividends and on long- and short-term capital gains. He finds evidence that taxes are capitalized into pre-tax asset prices to compensate investors for higher taxes. In a much earlier study, Poterba and Summers (1984) look at three different hypotheses about the effects of dividend taxation for Great Britain. Their findings do not support the tax capitalization view or the view that marginal investors remain untaxed. Instead they see the traditional view as more consistent with their results. In this view, dividend taxes reduce investment of the firm and lead to distortions in the allocation of capital. Sialm (2009) agrees to the tax capitalization and Poterba and Summers (1984) reject it. This suggests that empirical work on tax effects should be continued and extended to obtain a more general view from more time periods and markets about how taxes impact prices.

Economy-wide effects of taxes, such as effects on the risk-free rate and the equity premium are often studied in dynamic stochastic general equilibrium models. Santoro and Wei (2011) build on the work of Jermann (1998) and analyze corporate taxes and dividend taxes. They find corporate taxes do distort investment decisions but proportional taxes on dividends do not because dividends are taxed after investment decisions are made. They find an amplification mechanism through which corporate taxes increase reactions of consumption and investment to shocks. This can lead to a decrease in the risk-free rate and an increase in the equity premium. Croce et al. (2012) also concentrate on corporate taxes but additionally introduce tax rate uncertainty, which leads to significant risk premia in their model.

The literature, including the articles herein, separate cases in which cancelled debt is taxed and which it is not in such a way that only one case is applicable for a company and the other can never occur. However, for example in the U.S., a company may or may not be subject to a taxation of cancelled debt depending on the circumstances or state in which the company is in. That means that the cancelled-debt treatment can be refined. There may be defaults requiring cancelled debt to be taxed and other types of default when this is not true. Attaching probabilities to those cases, it is possible to combine them into a single valuation model.

With the introduction of earnings stripping rules (ESR) and thin cap rules, interest payments may not be fully deductible anymore. Those rules impose limits on how much interest can be deducted from taxable income. When this limit is exceeded, some part of the tax savings are lost and the value of tax savings shrink. The literature has treated ESRs for example in Förster et al. (2009), Eberl (2009), Maßbaum et al. (2012) and Buettner et al. (2012). There is still a lot of potential for continuing work on ESRs and thin cap rules. First, it can be unified with traditional valuations with risk-free and risky debt. Second, interest carry forward rules can be included in the analysis. Eventually, economy-wide effects of ESRs may be interesting for a closed but also for an open economy.

The study of different assets taxed differently is also interesting and valid for some markets.

6.2. Potential future research

For example, some assets may be exempt from certain kinds of taxes but others may not. Benninga and Sarig (2003) and Eikseth and Lindset (2009) provide some theoretical considerations on this. Redundant assets with different taxes may also be a good object of study for tax capitalization hypotheses.

The forth article is very theoretical in its nature. It is intended to widen one's horizon on the behavior of asset prices in certain economic models. Since it does not feature a representative agent, it also allows to study portfolio decisions in an equilibrium context. The economies presented are very simple endowment models regarding only two points in time, which is usually necessary when heterogeneous agents are regarded in incomplete markets. An idea to continue the analysis of portfolio decisions in multi-period models would be to look at complete market models as in Judd et al. (2003). The inclusion of taxes in this model may be a first step to an analysis of tax effects on portfolio decisions in a multi-period setting, general equilibrium and heterogeneous agents.

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