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Topological Index Criteria in DAE for Water Networks

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TOPOLOGICAL INDEX CRITERIA IN DAE FOR WATER NETWORKS

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ABSTRACT. The dynamics of pressurized water networks are naturally modeled by differential-algebraic equations (DAE). This article investigates fundamental structural properties of such a DAE model under weak regularity assumptions. The usual algebraic index-1 condition is shown to be necessary and sufficient with respect to several index concepts, as well as sufficient for solvability in a strong sense. Using the physical properties of nonlinear network elements and the inherent saddle point structure of network hydraulics, we then derive purely topological index criteria in terms of the network graph and the choice of control variables. Several examples illustrate the theoretical results and explore different non-index-1 situations. A brief discussion of the implications for operative planning based on discrete time DAE boundary value problems concludes the paper.

0. INTRODUCTION

Systems of differential-algebraic equations (DAE) are frequently involved in dynamic modeling of complex technical systems. This is particularly true for network type models where the mathematical equations are automatically generated from element submodels and connectivity relations. A key property characterizing the degree of difficulty of the theoretical analysis and numerical treatment of any DAE model is its index. Criteria that determine the index are well investigated for network models in several application areas, including rigid body systems in descriptor form [20, 27, 28], modified nodal analysis (MNA [10]) for electric circuits [9, 15, 29], and certain fields of chemical process engineering [23, 8]; for a survey see [18].

To the knowledge of the author, no comparable theory exists for pressurized water networks. The purpose of the present article is to fill this gap, with the aim of employing the structural results in the development of custom optimization algorithms for DAE boundary value problems arising in water management. Mathematical decision support tools, such as an optimization system used for operative planning by the municipal water supplier of Berlin [4, 5], are expected to benefit from the increased computational efficiency.

Although water networks share some similarities with electric circuits, existing index results for the latter are neither directly applicable nor easily adaptable. This is because water network models contain several node types in addition to several arc types, leading to more involved topological considerations. In particular, we need to analyze general submatrices of the incidence matrix, rather than just subsets of columns.

The paper is organized as follows. In Section 1 we introduce some notation and basic theoretical concepts. The hydraulic model of [4, 5] is presented in Section 2. Using the incidence matrix, the algebraic structure and solvability of the resulting DAE are analyzed. The main part, Section 3, provides an in-depth analysis of the DAE index. Several index concepts are addressed, and algebraic as well as topological index-1 conditions are derived. The theory is then explained by means of several examples in Section 4. A summary of the results with comments on their practical relevance concludes the paper.

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1. PRELIMINARIES

For the subsequent analysis, we need some standard function spaces, basic concepts of DAE theory, and the notion of a graph's incidence matrix.

1.1. **Function Spaces.** Given an interval $\mathcal{I} \subseteq \mathbb{R}$ and a normed space X, let $C^r(\mathcal{I}, X)$ denote the vector space of r times continuously differentiable functions $f\colon \mathcal{I} \to X$. If $\mathcal{I}$ is compact, then $C^r(\mathcal{I}, X)$ becomes itself a normed space under the natural norm

$$\|f\|_{C^r(\mathcal{I},X)} := \sum_{n=0}^{r} \|f^{(n)}\|_{C^0(\mathcal{I},X)}, \qquad \|f\|_{C^0(\mathcal{I},X)} := \max_{t\in\mathcal{I}} \|f(t)\|_X.$$

If $\mathcal{I}$ and X are clear from the context, we also write C^r for $C^r(\mathcal{I}, X)$ and $\|\cdot\|_\infty$ for $\|\cdot\|_{C^0}$. In writing $\|f^{(n)}\|_{C^0(\mathcal{I},X)}$, we use the fact that $X_0 := X$ with $\|\cdot\|_X$ is canonically isometric to the image spaces of the derivative mappings $f^{(n)}$ with their respective operator norms, $X_n := L(\mathbb{R}, X_{n-1})$, $n > 0$, and hence $\|\cdot\|_{C^0(\mathcal{I},X)}$ is defined on $C^{r-n}(\mathcal{I}, X_n) \subset C^0(\mathcal{I}, X)$. The canonical isometry $\iota_n\colon X_{n-1} \to X_n$ is defined by $\iota_n(A)t := tA$ for $t \in \mathbb{R}$.

1.2. **Differential-Algebraic Equations.** A differential-algebraic equation (DAE) is an equation of the general form

$$F(x'(t), x(t), t) = 0 \tag{1}$$

with $F_{x'} \equiv \partial F/\partial x'$ identically singular. In addition to the fully-implicit form (1), we will also consider DAE in semi-explicit form

$$A(x(t),t)x_1'(t) = f_1(x(t),t), \qquad x = (x_1, x_2), \tag{2a}$$

$$0 = f_2(x(t),t), \qquad A(x,t) \text{ nonsingular}. \tag{2b}$$

For further structural forms and special cases of (1) see, e.g., [3, §2].

In this article we adopt the technical setting of [22]. Assume that

- $F\colon \mathbb{R}^m \times \mathcal{D} \times \mathcal{I} \to \mathbb{R}^m$ is continuous with continuous partial derivatives $F_{x'}, F_x\colon \mathbb{R}^m \times \mathcal{D} \times \mathcal{I} \to \mathbb{R}^{m\times m}$, $\mathcal{D} \subseteq \mathbb{R}^m$ open, $\mathcal{I}$ an interval;
- the null space of $F_{x'}(y, x, t)$ is invariant for y and x,
$$\ker(F_{x'}(y,x,t)) = N(t), \quad (y,x,t) \in \mathbb{R}^m \times \mathcal{D} \times \mathcal{I};$$
- $N(t)$ varies smoothly with t, having a smooth projector function
$$Q \in C^1(\mathcal{I}, \mathbb{R}^{m\times m}), \quad Q(t)^2 = Q(t), \quad \operatorname{im}(Q(t)) = N(t), \quad t \in \mathcal{I}.$$

Defining the complementary projector function $P := I - Q$, it can then be shown [22] that

$$\begin{aligned} F(y,x,t) &= F(P(t)y, x, t), && (y,x,t) \in \mathbb{R}^m \times \mathcal{D} \times \mathcal{I}, \\ F(x'(t), x(t), t) &= F(P(t)x'(t), x(t), t) \\ &= F((Px)'(t) - P'(t)x(t), x(t), t), && x \in C^1(\mathcal{I}, \mathbb{R}^m). \end{aligned}$$

These facts are closely related to the natural geometric interpretation of (1) as a differential equation on a manifold [26, 24] wherein only the derivative $(Px)'$ appears but not $(Qx)'$. A natural solution space for the DAE (1) is therefore

$$C^1_P(\mathcal{I}, \mathbb{R}^m) := \{x \in C^0(\mathcal{I}, \mathbb{R}^m) \colon Px \in C^1(\mathcal{I}, \mathbb{R}^m)\},$$

equipped with the natural norm

$$\|x\|_{C^1_P} := \|x\|_\infty + \|(Px)'\|_\infty.$$

Considering the semi-explicit DAE (2) and defining $F := (-Ax_1' + f_1, f_2)$, the smoothness assumptions on F are satisfied iff $A \in C^0$ and $A, f_1, f_2 \in C^1$ with respect to the variables $x = (x_1, x_2) \in \mathbb{R}^{m_1+m_2}$. The null space assumptions are intrinsically satisfied since $N(t) \equiv \{0\} \times \mathbb{R}^{m_2}$, yielding

$$Q(t) \equiv \operatorname{Diag}(0, I), \quad P(t) \equiv \operatorname{Diag}(I, 0), \quad C^1_P = \{x \in C^0 \colon x_1 \in C^1\}.$$

The most important quantity characterizing the analytical and numerical properties of a DAE is its index. Loosely speaking, the index is a measure of difficulty in comparison to (possibly implicit) ordinary differential equations (ODE), which have index 0. There exist actually a plethora of index concepts; here we briefly recall only some of the most well-known definitions. (See, e.g., [7, 13] for comparisons of various indices.)

The concept of *differentiation index* [12, 3] is based on the idea of deriving an ODE for every component of x by differentiating (1) repeatedly with respect to time. To this end, observe that the derivatives of (1) can be written

$$0 = \frac{d^k}{dt^k} F(x', x, t) =: F_k(x^{(k+1)}, \dots, x^{(1)}, x, t) =: F_k(\mathbf{x}_{k+1}, x, t)$$

(provided that $F \in C^k$), and consider for $k \geq 0$ the *derivative array equations* [6],

$$0 = \mathbf{F}_k(\mathbf{x}_{k+1}, x, t) := \begin{pmatrix} F_0(\mathbf{x}_1, x, t) \\ \vdots \\ F_k(\mathbf{x}_{k+1}, x, t) \end{pmatrix}.$$

Definition 1 (Differentiation Index)**.** The *differentiation index* of the DAE (1) is the smallest integer $\nu_D \geq 0$ such that $\mathbf{F}_{\nu_D} = 0$ uniquely determines the variable x' as a continuous function of x and t.

The *perturbation index* [16, 17] measures the sensitivity of solutions of (1) to perturbations of the right-hand side.

Definition 2 (Perturbation Index)**.** The DAE (1) has *perturbation index* $\nu_P \geq 1$ along a solution x on $[a, b]$ if ν_P is the smallest integer such that for all functions $\tilde{x}$ having a defect

$$F(\tilde{x}', \tilde{x}, t) = \delta(t)$$

there exists on $[a, b]$ an estimate

$$\|\tilde{x}(t) - x(t)\| \leq c \left(\|\tilde{x}(0) - x(0)\| + \|\delta\|_{C^{(\nu_P - 1)}([a,t], \mathbb{R}^m)} \right)$$

whenever the expression in parentheses is sufficiently small.

The DAE (1) has *perturbation index zero* if there exists on $[a, b]$ an estimate

$$\|\tilde{x}(t) - x(t)\| \leq c \left(\|\tilde{x}(0) - x(0)\| + \max_{\tau \in [a,t]} \left\| \int_a^\tau \delta(s)\, ds \right\| \right).$$

In both cases, c denotes a constant that depends only on F, x, and the interval $[a, b]$.

The strongest index concept considered here is the *tractability index* ν_T [14, 21, 22], which is based on the relation of $N(t)$ with the space

$$S(y, x, t) := \{z \in \mathbb{R}^m : F_x(y, x, t)z \in \operatorname{im}(F_{x'}(y, x, t))\}, \quad (y, x, t) \in \mathbb{R}^m \times \mathcal{D} \times \mathcal{I}.$$

To avoid unnecessary technical complexity we define only the index-1 case; for generalizations see [21, 22].

Definition 3 (Tractability Index)**.** The DAE (1) is *index-1 tractable* (or *transferable*) on the open set $\mathcal{G} \subseteq \mathbb{R}^m \times \mathcal{D} \times \mathcal{I}$ if

$$N(t) \oplus S(y, x, t) = \mathbb{R}^m, \quad (y, x, t) \in \mathcal{G},$$

that is, $N(t) + S(y, x, t) = \mathbb{R}^m$ and $N(t) \cap S(y, x, t) = \{0\}$.

1.3. **Incidence Matrix.** The topology of a directed graph is conveniently represented by its *incidence matrix*. We recall this concept and summarize some well-known structural properties without proofs; the interested reader is referred to [11, 1]. Only simple graphs will be considered throughout this paper, and the arc directions have no graph-theoretic relevance; they only specify positive directions of flow.

Definition 4. The *node-arc incidence matrix* of a directed graph $G = (\mathcal{N}, \mathcal{A})$ is

$$E(G) \in \mathbb{R}^{|\mathcal{N}|\times|\mathcal{A}|} \quad \text{where} \quad E_{ia}(G) := \begin{cases} -1 & \text{if node } i \text{ is the tail of arc } a, \\ 0 & \text{if node } i \text{ is not incident to } a, \\ +1 & \text{if node } i \text{ is the head of arc } a. \end{cases}$$

Proposition 1. *Let* G *consist of connected components* $G_1, \dots, G_n$, $n \geq 1$. *Then:*
(1) Every column of $E(G)$ *contains precisely two entries:* -1 *and* $+1$.
(2) The rows of $E(G)$ *sum up to zero.*
(3) G *has no isolated nodes* $\iff$ $E(G)$ *has no zero rows.*
(4) G *is a tree* $\iff$ $E(G)$ *has rank* $|\mathcal{A}| = |\mathcal{N}| - 1$.
(5) G *is connected* $\iff$ $E(G)$ *has rank* $|\mathcal{N}| - 1$.
(6) G *is connected* $\iff$ *every proper row subset of* $E(G)$ *has full rank.*
(7) $E(G) = \mathrm{Diag}(\{E(G_\nu)\})$.
(8) $\mathrm{rank}(E(G)) = \sum_\nu \mathrm{rank}(E(G_\nu)) = |\mathcal{N}| - n$.
(9) G *is acyclic (i.e., a forest)* $\iff$ $E(G)$ *has full column rank.*

2. Network Dynamics

2.1. **Hydraulic Model.** Our network model [4] is based on a directed graph $G = (\mathcal{N}, \mathcal{A})$ whose node set consists of reservoirs, junctions, and tanks,

$$\mathcal{N} = \mathcal{N}_{\mathrm{rs}} \cup \mathcal{N}_{\mathrm{jc}} \cup \mathcal{N}_{\mathrm{tk}},$$

and whose arc set consists of pipes, pumps, and valves,

$$\mathcal{A} = \mathcal{A}_{\mathrm{pi}} \cup \mathcal{A}_{\mathrm{pu}} \cup \mathcal{A}_{\mathrm{vl}}.$$

The network behavior will be studied over some closed time interval $\bar{\mathcal{I}} \subseteq \mathbb{R}$, $\mathcal{I}$ open, where the case $\mathcal{I} = \mathbb{R}$ is explicitly allowed. Dynamic variables on $\bar{\mathcal{I}}$ include the node pressures H_j (potential heads), arc flows Q_{ij} (volumetric flowrates), and pressure differences ΔH_{ij} across control elements:

$$\begin{aligned} H &= (H_{\mathrm{rs}}, H_{\mathrm{jc}}, H_{\mathrm{tk}}) = (H_j)_{j\in\mathcal{N}}, \\ Q &= (Q_{\mathrm{pi}}, Q_{\mathrm{pu}}, Q_{\mathrm{vl}}) = (Q_{ij})_{ij\in\mathcal{A}}, \\ \Delta H &= (\Delta H_{\mathrm{pu}}, \Delta H_{\mathrm{vl}}) = (\Delta H_{ij})_{ij\in\mathcal{A}_{\mathrm{pu}}\cup\mathcal{A}_{\mathrm{vl}}}. \end{aligned}$$

Arc directions indicate the positive direction of flow. The hydraulic model governing the physical network behavior includes given reservoir heads, flow balance equations at all other nodes, and pressure difference equations along arcs:

$$\begin{aligned} H_j &= \bar{H}_j, && j \in \mathcal{N}_{\mathrm{rs}}, \\ \sum_{i:\, ij\in\mathcal{A}} Q_{ij} - \sum_{k:\, jk\in\mathcal{A}} Q_{jk} &= D_j, && j \in \mathcal{N}_{\mathrm{jc}}, \\ \sum_{i:\, ij\in\mathcal{A}} Q_{ij} - \sum_{k:\, jk\in\mathcal{A}} Q_{jk} &= A_j(H_j)H_j', && j \in \mathcal{N}_{\mathrm{tk}}, \\ H_j - H_i &= -\varphi_{ij}(Q_{ij}), && ij \in \mathcal{A}_{\mathrm{pi}}, \\ H_j - H_i &= +\Delta H_{ij}, && ij \in \mathcal{A}_{\mathrm{pu}}, \\ H_j - H_i &= -\Delta H_{ij}, && ij \in \mathcal{A}_{\mathrm{vl}}. \end{aligned}$$

We seek continuous solutions $H_j, Q_{ij}, \Delta H_{ij} \in C^0(\bar{\mathcal{I}}, \mathbb{R})$, with the tank filling levels being continuously differentiable in addition, $H_j \in C^1(\bar{\mathcal{I}}, \mathbb{R})$ for $j \in \mathcal{N}_{\mathrm{tk}}$.

Externally given data include: $\bar{H}_j(t)$, the water levels at reservoirs, $D_j(t)$, the predicted demands at junctions, $A_j(H_j)$, the effective cross-sectional tank areas, and $\varphi_{ij}(Q_{ij})$, the approximate hydraulic pressure losses along pipe segments. (Details of the model are given in [4]. There we also allow a piecewise continuous time-dependence of A_j.)

2.2. **General Assumptions.** Throughout this paper we impose the following topological assumptions on the network graph:

(T1) G is connected.

(T2) G contains at least one reservoir or tank: $\mathcal{N}_{\mathrm{rs}} \cup \mathcal{N}_{\mathrm{tk}} \neq \emptyset$.

In addition, suitable properties are assumed of the data functions:

(D1) $\forall j \in \mathcal{N}_{\mathrm{rs}}$: $\bar{H}_j \in C^0(\bar{\mathcal{I}}, \mathbb{R})$.

(D2) $\forall j \in \mathcal{N}_{\mathrm{jc}}$: $D_j \in C^0(\bar{\mathcal{I}}, \mathbb{R})$.

(D3) $\forall j \in \mathcal{N}_{\mathrm{tk}}$: $A_j \in C^0(\mathbb{R}, \mathbb{R})$ is strictly positive (i.e., $A_j \geq \delta_{\mathrm{tk}} > 0$ on $\mathbb{R}$) and locally Lipschitz continuous.

(D4) $\forall ij \in \mathcal{A}_{\mathrm{pi}}$: $\varphi_{ij} \in C^1(\mathbb{R}, \mathbb{R})$ is odd and strictly growing (i.e., $\varphi'_{ij} \geq \delta_{\mathrm{pi}} > 0$ on $\mathbb{R}$).

2.3. **Model Structure.** For a concise formulation, we use the node-arc incidence matrix of the network graph G, partitioned like H and Q by node and arc types:

$$E(G) = E = \begin{pmatrix} E_{\mathrm{rs}} \\ E_{\mathrm{jc}} \\ E_{\mathrm{tk}} \end{pmatrix} = \begin{pmatrix} E_{\mathrm{pi}} & E_{\mathrm{pu}} & E_{\mathrm{vl}} \end{pmatrix} = \begin{pmatrix} E_{\mathrm{rs,pi}} & E_{\mathrm{rs,pu}} & E_{\mathrm{rs,vl}} \\ E_{\mathrm{jc,pi}} & E_{\mathrm{jc,pu}} & E_{\mathrm{jc,vl}} \\ E_{\mathrm{tk,pi}} & E_{\mathrm{tk,pu}} & E_{\mathrm{tk,vl}} \end{pmatrix}.$$

The above hydraulic model can then be written as

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ A_{\mathrm{tk}}(H_{\mathrm{tk}})H'_{\mathrm{tk}} \end{pmatrix} = \begin{pmatrix} \varphi_{\mathrm{pi}}(Q_{\mathrm{pi}}) + E^*_{\mathrm{pi}}H \\ E^*_{\mathrm{pu}}H - \Delta H_{\mathrm{pu}} \\ E^*_{\mathrm{vl}}H + \Delta H_{\mathrm{vl}} \\ H_{\mathrm{rs}} \\ E_{\mathrm{jc}}Q \\ E_{\mathrm{tk}}Q \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{H}_{\mathrm{rs}} \\ D_{\mathrm{jc}} \\ 0 \end{pmatrix} \tag{3}$$

where the data are defined as

$$A_{\mathrm{tk}}(H_{\mathrm{tk}}) := \mathrm{Diag}\big(A_j(H_j)\big)_{j\in\mathcal{N}_{\mathrm{tk}}}, \qquad \bar{H}_{\mathrm{rs}} := \big(\bar{H}_j\big)_{j\in\mathcal{N}_{\mathrm{rs}}},$$
$$\varphi_{\mathrm{pi}}(Q_{\mathrm{pi}}) := \big(\varphi_{ij}(Q_{ij})\big)_{ij\in\mathcal{A}_{\mathrm{pi}}}, \qquad D_{\mathrm{jc}} := \big(D_j\big)_{j\in\mathcal{N}_{\mathrm{jc}}}.$$

By assumption (D3), the diagonal matrix $A_{\mathrm{tk}}(H_{\mathrm{tk}})$ is positive definite on $\mathbb{R}^{|\mathcal{N}_{\mathrm{tk}}|}$ and has a uniformly bounded inverse, $A_{\mathrm{tk}}(H_{\mathrm{tk}})^{-1} \leq \delta_{\mathrm{tk}}^{-1} I$.

A central object in the analysis of the model (3) is the Wronskian of the right-hand side,

$$\overline{W}(Q_{\mathrm{pi}}) := \left(\begin{array}{ccc|ccc|cc} \varphi'_{\mathrm{pi}}(Q_{\mathrm{pi}}) & & & E^*_{\mathrm{rs,pi}} & E^*_{\mathrm{jc,pi}} & E^*_{\mathrm{tk,pi}} & 0 & 0 \\ & 0 & & E^*_{\mathrm{rs,pu}} & E^*_{\mathrm{jc,pu}} & E^*_{\mathrm{tk,pu}} & -I & \\ & & 0 & E^*_{\mathrm{rs,vl}} & E^*_{\mathrm{jc,vl}} & E^*_{\mathrm{tk,vl}} & & +I \\ \hline 0 & 0 & 0 & I & & & & \\ E_{\mathrm{jc,pi}} & E_{\mathrm{jc,pu}} & E_{\mathrm{jc,vl}} & & 0 & & & \\ E_{\mathrm{tk,pi}} & E_{\mathrm{tk,pu}} & E_{\mathrm{tk,vl}} & & & 0 & & \end{array}\right). \tag{4}$$

Note that most of the entries are unity (± 1). Non-unit entries appear only in the leading diagonal block which, by assumption (D4), is positive definite on $\mathbb{R}^{|\mathcal{A}_{\mathrm{pi}}|}$ with uniformly bounded inverse,

$$\varphi'_{\mathrm{pi}}(Q_{\mathrm{pi}}) = \mathrm{Diag}\big(\varphi'_{ij}(Q_{ij})\big)_{ij\in\mathcal{A}_{\mathrm{pi}}} \geq \delta_{\mathrm{pi}} I, \qquad \varphi'_{\mathrm{pi}}(Q_{\mathrm{pi}})^{-1} \leq \delta_{\mathrm{pi}}^{-1} I.$$

The entry positions as well as their signs are therefore *structural* properties of the model. Note also that the left six by six block of the Wronskian exhibits an almost perfect primal-dual (saddle point, or KKT type) structure where Q corresponds to primal variables and H corresponds to dual variables, with the exception of the prescribed reservoir heads H_{rs}.

2.3.1. *Algebraic Variables and Control Variables.* Each control element (pump or valve) has two dynamic variables: flow Q_{ij} and pressure difference ΔH_{ij}. To simulate the model, either Q_{ij} or ΔH_{ij} may be selected as the control variable (to be specified as model input); the other quantity then becomes an algebraic variable (implicitly determined by the model). In effect, this yields partitionings of the sets of pumps and valves into flow-controlled and pressure-controlled elements,

$$\mathcal{A}_{pu} = \mathcal{A}_{pu}^{Q} \cup \mathcal{A}_{pu}^{H}, \qquad \mathcal{A}_{vl} = \mathcal{A}_{vl}^{Q} \cup \mathcal{A}_{vl}^{H}. \tag{5}$$

The associated vector components of Q and ΔH are picked by selection matrices having a single unit entry per row,

$$\begin{aligned} \Sigma_{pu}^{Q} &\in \mathbb{R}^{|\mathcal{A}_{pu}^{Q}|\times|\mathcal{A}_{pu}|}, & \Sigma_{vl}^{Q} &\in \mathbb{R}^{|\mathcal{A}_{vl}^{Q}|\times|\mathcal{A}_{vl}|}, \\ \Sigma_{pu}^{H} &\in \mathbb{R}^{|\mathcal{A}_{pu}^{H}|\times|\mathcal{A}_{pu}|}, & \Sigma_{vl}^{H} &\in \mathbb{R}^{|\mathcal{A}_{vl}^{H}|\times|\mathcal{A}_{vl}|}. \end{aligned}$$

Defining corresponding orthogonal projection matrices $\Pi_{pu}^{Q} := (\Sigma_{pu}^{Q})^{*}(\Sigma_{pu}^{Q})$ etc.,

$$\Pi_{pu}^{Q}, \Pi_{pu}^{H} \in \mathbb{R}^{|\mathcal{A}_{pu}|\times|\mathcal{A}_{pu}|}, \qquad \Pi_{vl}^{Q}, \Pi_{vl}^{H} \in \mathbb{R}^{|\mathcal{A}_{vl}|\times|\mathcal{A}_{vl}|},$$

control variables u_{pu}, u_{vl} and algebraic variables x_{pu}, x_{vl} of pumps and valves are then obtained by swapping elements between subvectors (Q_{pu}, Q_{vl}) and $(\Delta H_{pu}, \Delta H_{vl})$,

$$\begin{aligned} u_{pu} &:= \Pi_{pu}^{Q} Q_{pu} + \Pi_{pu}^{H} \Delta H_{pu}, & x_{pu} &:= \Pi_{pu}^{H} Q_{pu} + \Pi_{pu}^{Q} \Delta H_{pu}, \\ u_{vl} &:= \Pi_{vl}^{Q} Q_{vl} + \Pi_{vl}^{H} \Delta H_{vl}, & x_{vl} &:= \Pi_{vl}^{H} Q_{vl} + \Pi_{vl}^{Q} \Delta H_{vl}. \end{aligned}$$

This yields a splitting of $(Q, H, \Delta H)$ into differential variables x_1, algebraic variables x_2, and control variables u:

$$\begin{aligned} x_1 &:= H_{tk}, & m_1 &= |\mathcal{N}_{tk}|, \\ x_2 &:= \big(Q_{pi}, x_{pu}, x_{vl}, H_{rs}, H_{jc}\big), & m_2 &= |\mathcal{A}| + |\mathcal{N}_{rs} \cup \mathcal{N}_{jc}|, \\ u &:= \big(u_{pu}, u_{vl}\big), & m_u &= |\mathcal{A}_{pu} \cup \mathcal{A}_{vl}|. \end{aligned}$$

With rows and columns permuted into variable order (x_1, x_2, u), the Wronskian becomes

$$W(Q_{pi}) = \left(\begin{array}{c|ccccc|cc} 0 & E_{tk,pi} & E_{tk,pu}\Pi_{pu}^{H} & E_{tk,vl}\Pi_{vl}^{H} & 0 & 0 & E_{tk,pu}\Pi_{pu}^{Q} & E_{tk,vl}\Pi_{vl}^{Q} \\ \hline E_{tk,pi}^{*} & \varphi_{pi}'(Q_{pi}) & & & E_{rs,pi}^{*} & E_{jc,pi}^{*} & 0 & 0 \\ E_{tk,pu}^{*} & & -\Pi_{pu}^{Q} & & E_{rs,pu}^{*} & E_{jc,pu}^{*} & -\Pi_{pu}^{H} & \\ E_{tk,vl}^{*} & & & +\Pi_{vl}^{Q} & E_{rs,vl}^{*} & E_{jc,vl}^{*} & & +\Pi_{vl}^{H} \\ 0 & 0 & 0 & 0 & I & & 0 & 0 \\ 0 & E_{jc,pi} & E_{jc,pu}\Pi_{pu}^{H} & E_{jc,vl}\Pi_{vl}^{H} & & 0 & E_{jc,pu}\Pi_{pu}^{Q} & E_{jc,vl}\Pi_{vl}^{Q} \end{array}\right).$$

2.3.2. *DAE Formulation.* By selecting control variables, our hydraulic model receives the structure of a semi-explicit DAE over the network graph G,

$$A_{tk}(x_1)x_1' = f_1(x_1, x_2, t), \tag{6a}$$

$$0 = f_2(x_1, x_2, t), \tag{6b}$$

where the control dependence of f_1, f_2 is hidden in the explicit time dependence,

$$f_i(x_1, x_2, t) = \bar{f}_i(x_1, x_2, u(t), t), \qquad (x_1, x_2, t) \in \mathbb{R}^{m_1+m_2} \times \bar{\mathcal{J}}.$$

General Assumption. Throughout the paper the control is continuous: $u \in C^0(\bar{\mathcal{J}}, \mathbb{R}^{m_u})$.

Proposition 2 (DAE structure). *Consider the hydraulic model* (3) *in DAE form* (6). *Then the following hold:*

(i) *The functions* f_1, f_2 *admit decompositions*

$$\begin{aligned} f_1(x_1, x_2, t) &= \qquad\qquad L_{12}x_2 + \qquad\qquad f_{1t}(t), \\ f_2(x_1, x_2, t) &= L_{21}x_1 + L_{22}x_2 + f_{pi}(Q_{pi}) + f_{2t}(t), \end{aligned}$$

$$f_{it}(t) = L_{iu}u(t) + f_{it}^0(t), \quad i = 1, 2,$$

where $L_{ij} \in \mathbb{R}^{m_i \times m_j}$, $L_{iu} \in \mathbb{R}^{m_i \times m_u}$, $f_{pi} \in C^1(\mathbb{R}^{|\mathcal{A}_{pi}|}, \mathbb{R}^{m_2})$, $f_{it}^0 \in C^0(\bar{\mathcal{I}}, \mathbb{R}^{m_i})$.

(ii) *The partial derivative* $\partial f_2/\partial x_2$ *admits the corresponding decomposition*

$$\frac{\partial f_2}{\partial x_2}(x_1, x_2, t) = f_{22}'(x_2) = L_{22} + f_{pi}'(Q_{pi}) =: W_{22}(Q_{pi})$$

where $f_{22}(x_2) := L_{22}x_2 + f_{pi}(Q_{pi})$ *and* $W_{22} \in C^0(\mathbb{R}^{|\mathcal{A}_{pi}|}, \mathbb{R}^{m_2 \times m_2})$.

Proof. Since Q_{pi} belongs entirely to x_2 and $x_1 = H_{tk}$, the claims are immediate from (3) and the structure of $W(Q_{pi})$ under continuity of u and assumptions (D1), (D2), (D4). The six superblocks of $W(Q_{pi})$ correspond to L_{11}, L_{21}, L_{12}, $W_{22}(Q_{pi})$, L_{1u}, and L_{2u}. □

Remark 1. Item (i) implies that (6) satisfies the structural assumptions of Section 1.2, except that we only need local Lipschitz continuity of A_{tk} rather than $A_{tk} \in C^1$.

As a first consequence of the decomposability, we find by the following auxiliary result that $W_{22}(Q_{pi})$ has constant rank.

Proposition 3. *Consider a mapping* $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *with a linear part,*

$$f(x) = \begin{pmatrix} f_1(x) \\ A_2 x \end{pmatrix} \in \mathbb{R}^{n_1 + n_2}.$$

Suppose that $f_1(0) = 0$ *and that* f_1' *has full rank on* $\ker(A_2)$, $\operatorname{rank}(f_1'(x)) \equiv n_1$. *Then* f' *has constant rank on* $\mathbb{R}^n$, $\operatorname{rank}(f'(x)) \equiv n - k$, *and* $f^{-1}(\{0\})$ *is a* k*-dimensional submanifold of* $\mathbb{R}^n$, *where* $k := \dim\ker(A_2) - n_1 \equiv \dim\ker(f'(x))$.

Proof. Clearly, $f^{-1}(\{0\}) = \{x \in \ker(A_2) \colon f_1(x) = 0\}$. Since $f_1'(x) \in \mathbb{R}^{n_1 \times n}$ is surjective for $x \in \ker(A_2)$ and $f_1(0) = 0$, $f^{-1}(\{0\})$ is an n_1-codimensional submanifold of $\ker(A_2)$, and hence a k-dimensional submanifold of $\mathbb{R}^n$. It follows that f' has constant rank. (See, e.g., [19, Thm. 1.3.8] with $M := \ker(A_2)$, $N := \mathbb{R}^n$, $N_1 := \{0\}$, and $F := f_1$.) □

Corollary 1. *The rank of* $W_{22}(Q_{pi})$ *is constant on* $\mathbb{R}^{|\mathcal{A}_{pi}|}$.

Proof. Apply Proposition 3 to $f_{22} \in C^1(\mathbb{R}^{m_2}, \mathbb{R}^{m_2})$. Observe that only the pipe flow components are nonlinear, that $\varphi_{pi}(0) = 0$ since φ_{pi} is odd, and that the derivative of the nonlinear part has full rank on all of $\mathbb{R}^n$ since $\varphi_{pi}'(Q_{pi}) \geq \delta_{pi} I$. □

2.3.3. *DAE Solvability.* Under the nice structure just established, we find that invertibility of W_{22} is a sufficient condition for the global existence and uniqueness of solutions.

Theorem 1. *Consider again the hydraulic model* (3) *in DAE form* (6). *If* $W_{22}(Q_{pi})$ *is nonsingular on* $\mathbb{R}^{|\mathcal{A}_{pi}|}$, *then the following hold:*

(i) *One can globally solve* $f_2 = 0$ *for* x_2 *as a continuous function* $\psi(x_1, t)$ *that is* C^1 *with respect to* x_1*:*

$$f_2(x_1, x_2, t) = 0 \iff x_2 = \psi(x_1, t), \quad (x_1, x_2, t) \in \mathbb{R}^{m_1 + m_2} \times \bar{\mathcal{I}}.$$

(ii) *Given any* $(x_1^0, t^0) \in \mathbb{R}^{m_1} \times \bar{\mathcal{I}}$, *the DAE has a unique* C_p^1*-solution* x *on* $\bar{\mathcal{I}}$ *satisfying* $x(t^0) = x^0$ *with* $x_2^0 := \psi(x_1^0, t^0)$.

(iii) The mapping $\psi \in C^0(\mathbb{R}^{m_1} \times \mathcal{I}, \mathbb{R}^{m_2})$ *defines an* (m_1+1)*-dimensional* C^0*-submanifold of* $\mathbb{R}^m \times \mathcal{I}$ *consisting of the DAE solutions,*

$$M := \{(x_1, \psi(x_1,t), t) \colon (x_1,t) \in \mathbb{R}^{m_1} \times \mathcal{I}\}.$$

Moreover, the following projections are m_1*-dimensional* C^1*-submanifolds of* $\mathbb{R}^m$*:*

$$M_t := \{(x_1, \psi(x_1,t)) \colon x_1 \in \mathbb{R}^{m_1}\}, \quad t \in \mathcal{I}.$$

Proof. Since $W_{22} \in C^0$ is nonsingular on $\mathbb{R}^{|\mathcal{A}_{\text{pi}}|}$, the inverse $(f'_{22})^{-1}$ is continuous and hence locally bounded on $\mathbb{R}^{m_2}$. It follows that f_{22} is everywhere locally invertible and hence globally invertible, $f_{22}^{-1} \in C^1(\mathbb{R}^{m_2}, \mathbb{R}^{m_2})$. Then $\psi(x_1,t) := f_{22}^{-1}(-L_{21}x_1 - f_{2t}(t))$ has the stated properties, and the C^1_P-solutions of (6) have the form $(x_1(t), \psi(x_1(t),t))$ where x_1 is a solution of the underlying explicit ODE:

$$x_1' = A_{\text{tk}}(x_1)^{-1} f_1(x_1, \psi(x_1,t), t), \quad t \in \bar{\mathcal{I}}. \tag{7}$$

The right-hand side of (7) is continuous on $\mathbb{R}^{m_1} \times \bar{\mathcal{I}}$ and, by the following proposition, locally Lipschitz continuous with respect to x_1. The theorem of Picard–Lindelöf together with the continuation theorem thus guarantee the global existence and uniqueness of ODE solutions $x_1 \in C^1(\bar{\mathcal{I}}, \mathbb{R}^{m_1})$, which implies the remaining statements. □

Proposition 4. *Under the hypothesis of Theorem* 1*, the right-hand side of* (7) *is locally Lipschitz continuous with respect to* x_1.

Proof. For $(x,t) \in \mathbb{R}^{m_1} \times \bar{\mathcal{I}}$, let $V(x) := A_{\text{tk}}(x)^{-1}$ and $g(x,t) := f_1(x, \psi(x,t), t)$. The right-hand side of (7) then reads $f(x,t) := V(x)g(x,t)$. For the maximum norm on $\mathbb{R}^{m_1}$ we obtain the estimate

$$\begin{aligned}
\|f(x,t) - f(y,t)\| &= \|V(x)[g(x,t) - g(y,t)] + [V(x) - V(y)]g(y,t)\| \\
&\le \|V(x)\| \, \|g(x,t) - g(y,t)\| + \|V(x) - V(y)\| \, \|g(y,t)\| \\
&\le \delta_{\text{tk}}^{-1} \|g(x,t) - g(y,t)\| + \delta_{\text{tk}}^{-2} \|A_{\text{tk}}(x) - A_{\text{tk}}(y)\| \, \|g(y,t)\|.
\end{aligned}$$

Here we have used that $|a^{-1} - b^{-1}| = |a-b|/|ab| \le \delta^{-2}|a-b|$ for all $a, b \in \mathbb{R}_{\ge\delta}$. Since A_{tk} is locally Lipschitz continuous and g is locally bounded, it remains to show that g is locally Lipschitz continuous with respect to x. Statement (i) of Proposition 2 yields

$$\|g(x,t) - g(y,t)\| = \|L_{12}[\psi(x,t) - \psi(y,t)]\| \le \|L_{12}\| \, \|\psi(x,t) - \psi(y,t)\|.$$

Since ψ is C^1 with respect to x_1, ψ and thus g are locally Lipschitz continuous. □

Corollary 2. *Under the hypothesis of Theorem* 1*, the DAE* (6) *is solvable on* $\mathbb{R}^{2m} \times \mathcal{I}$ *in the standard sense* [3, Def. 2.2.1]*, except that we only require* $x \in C^1_P$ *rather than* $x \in C^1$.

Proof. Fix some $t_0 \in \mathcal{I}$ and let $X_1(c,t)$ denote the unique solution of ODE (7) with initial value $x_1(t_0) = c$. Then Definition 2.2.1 in [3] is satisfied with $\Omega := \mathbb{R}^{2m} \times \mathcal{I}$, $r := m_1$, $\tilde{\Omega} := \mathbb{R}^r$, and $\phi(t,c) := (X_1(c,t), \psi(X_1(c,t),t))$ for $(c,t) \in \tilde{\Omega} \times \mathcal{I}$. That is, DAE (6) has an m_1-dimensional family of solutions $\phi(t,c)$, parameterized by $c \in \mathbb{R}^{m_1}$. □

Remark 2. It is easily verified that the stronger regularity assumptions of [3, Def. 2.2.1] will be satisfied in our model if we require all data to be continuously differentiable:

$$D_{\text{jc}}, \bar{H}_{\text{rs}}, A_{\text{tk}}, u \in C^1. \tag{*}$$

This implies that the mapping F and the solutions x will also be continuously differentiable.

Remark 3. Solvability is also known as *regularity* [25] or *geometric solvability* [7, Def. 1]. The DAE (6) is actually *uniformly 1-solvable* in the sense of [7, Def. 2] (with $\varepsilon = \infty$): $F(x'(t), x(t), t) = \delta(t)$ remains solvable for arbitrary defects $\delta \in C^0(\bar{\mathcal{I}}, \mathbb{R}^m)$, and the solutions $x \in C^1_P$ depend continuously on $\delta \in C^0$. To verify this, we just have to redefine $\psi(x_1,t) := f_{22}^{-1}(\delta_2(t) - L_{21}x_1 - f_{2t}(t))$ and $x_1' = A_{\text{tk}}(x_1)^{-1}[f_1(x_1, \psi(x_1,t), t) - \delta_1(t)]$ in the proof of Theorem 1. Again, the stronger assumptions (*) are originally required.

3. DAE INDEX

In the following we study the index of the DAE (6). Although we consider several index concepts, they all coincide in the nice situation of Theorem 1, where the DAE solutions for arbitrary forcing functions define a regular flow on a manifold M: the sufficient condition for solvability turns out to be necessary and sufficient for index-1. We formally state the identity for the index concepts defined above, and drop the distinction from then on.

Theorem 2. *The invertibility of $\partial f_2/\partial x_2 = W_{22}$ is necessary and sufficient for the DAE* (6) *to be index-1 on $\mathbb{R}^m \times \bar{\mathcal{I}}$ according to each of the three concepts above:*

$$\nu_D = \nu_P = \nu_T = 1.$$

Proof. (a) Differentiation index: for semi-explicit DAE it is well-known that $\nu_D = 1$ if and only if W_{22} is nonsingular. Briefly, F_0 uniquely determines x_1' (by the differential part) but not x_2'. The latter is determined by F_1 (the algebraic part) iff W_{22} is nonsingular.

(b) Tractability index: to see why $\nu_T = 1$ we use the fact [22, §3] that transferability is equivalent to nonsingularity of the matrix

$$G_1(y, x, t) := F_{x'}(y, x, t) + F_x(y, x, t)Q(t).$$

In the DAE (6) this is equivalent to nonsingularity of W_{22} since

$$\begin{aligned} G_1(y,x,t) &= \begin{pmatrix} -A_{tk}(x_1) & \\ & 0 \end{pmatrix} + \begin{pmatrix} * & \frac{\partial f_1}{\partial x_2}(x_1, x_2, t) \\ * & \frac{\partial f_2}{\partial x_2}(x_1, x_2, t) \end{pmatrix} \begin{pmatrix} 0 & \\ & I \end{pmatrix} \\ &= \begin{pmatrix} -A_{tk}(x_1) & L_{12} \\ 0 & W_{22}(Q_{pi}) \end{pmatrix}. \end{aligned}$$

(c) Perturbation index: nonsingularity of W_{22} is a sufficient condition since any semi-explicit DAE with $\nu_D = 1$ and any transferable DAE also has $\nu_P = 1$ along any solution $x \colon [a, b] \to \mathbb{R}^m$; see, e.g., [17, §VII.1] and [22, Thm. 3.1]. It remains to show that the condition is also necessary.

Assume that W_{22} is singular. If the DAE has no solution, nothing needs to be proved. Otherwise, given any solution $x \colon [a, b] \to \mathbb{R}^m$ and constants $c > 0$, $\epsilon > 0$, we construct a function $\tilde{x} := x + z$ and select $t_* \in [a, b]$ such that

$$\|\delta\|_{C^0([a,b],\mathbb{R}^m)} < \epsilon, \qquad \|z(t_*)\| > c\left(\|z(a)\| + \|\delta\|_{C^0([a,t_*],\mathbb{R}^m)}\right),$$

where $\delta(t) := F(\tilde{x}'(t), \tilde{x}(t), t)$. Using $\tilde{x} = x + z$ and $F(x', x, t) = 0$, Proposition 2 yields

$$\delta = \begin{pmatrix} L_{12}z_2 + A_{tk}(x_1)x_1' - A_{tk}(x_1 + z_1)(x_1' + z_1') \\ L_{21}z_1 + L_{22}z_2 - f_{pi}(Q_{pi}) + f_{pi}(Q_{pi} + z_{pi}) \end{pmatrix} =: \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix},$$

where z_{pi} denotes the pipe flow component of z. Define $g_2 \in C^1(\mathbb{R}^{m_2}, \mathbb{R}^{m_2})$ by

$$g_2(z_2) := L_{22}z_2 + f_{pi}(Q_{pi} + z_{pi}) - f_{pi}(Q_{pi}).$$

Then we have $g_2(0) = 0$ and $g_2'(z_2) = W_{22}(Q_{pi} + z_{pi})$. By Proposition 3, $Z := g_2^{-1}(\{0\})$ is a C^1-submanifold in $\mathbb{R}^{m_2}$ of dimension $\dim\ker(W_{22}) > 0$. For $\gamma > 0$ sufficiently small we can thus find a curve $z_2 \in C^1([a, b], Z)$ satisfying $\|z_2(t)\|_\infty = \gamma(t - a)$. Next, choose z_1 such that δ_1 vanishes,

$$z_1(t) := \int_a^t \left[A_{tk}(x_1(\tau) + z_1(\tau))^{-1}[L_{12}z_2(\tau) + A_{tk}(x_1(\tau))x_1'(\tau)] - x_1'(\tau)\right] d\tau. \tag{8}$$

These choices imply $z(a) = 0$ and $\delta = (0, \delta_2)$ where $\delta_2(t) = L_{21}z_1(t)$. Denote by L the Lipschitz constant of A_{tk} on a compact set containing $x_1([a, b]) \cup \tilde{x}_1([a, b])$, observe that

$$A_{tk}(x_1 + z_1)^{-1}A_{tk}(x_1)x_1' - x_1' = A_{tk}(x_1 + z_1)^{-1}[A_{tk}(x_1) - A_{tk}(x_1 + z_1)]x_1',$$

and define

$$\alpha := \delta_{tk}^{-1}\|L_{12}\|\gamma, \qquad \beta := \delta_{tk}^{-1}L\|x_1'\|_{C^0([a,b],\mathbb{R}^{m_1})}.$$

(Here $\|\cdot\|$ again denotes the maximum norm on $\mathbb{R}^m$, $\mathbb{R}^{m_1}, \dots$). Then we obtain

$$\|z_1(t)\| \le \int_a^t \left[\alpha(s-a) + \beta\|z_1(s)\|\right] ds = \frac{\alpha}{2}(t-a)^2 + \beta \int_a^t \|z_1(s)\|\, ds. \tag{9}$$

Case 1: If $m_1 = 0$, we have $\delta = \delta_2 = 0$ and $z = z_2$ on $[a, b]$, hence $\|\delta\|_{C^0} = 0 < \epsilon$ and $\|z(t_*)\| = (t_* - a)\gamma > c(0 + 0)$ for $t_* > a$.

Case 2: If $m_1 \ne 0$ and $L_{12} = 0$, we have $z_1 = 0$ on $[a, b]$ by (8) since $z_1(a) = 0$. This yields $\delta = 0$ and $z = (0, z_2)$ on $[a, b]$, hence the same inequalities as in case 1.

Case 3: If $m_1 \ne 0$, $L_{12} \ne 0$, and $x_1' = 0$ on $[a, b]$, we have $\alpha > 0$ and $\beta = 0$, and (9) simplifies to $\|z_1(t)\| \le \frac{\alpha}{2}(t-a)^2 =: \alpha\zeta(t)$. The proof continues after case 4.

Case 4: Otherwise we have $\alpha > 0$ and $\beta > 0$. Apply the Lemma of Gronwall and then a series of algebraic transformations to derive

$$\begin{aligned}
\|z_1(t)\| &\le \frac{\alpha}{2}(t-a)^2 + \beta \int_a^t e^{\beta(t-s)} \frac{\alpha}{2}(s-a)^2\, ds \\
&= \frac{\alpha}{\beta^2}\left(e^{\beta(t-a)} - \beta(t-a) - 1\right) = \alpha \sum_{k=2}^{\infty} \frac{\beta^{k-2}}{k!}(t-a)^k =: \alpha\zeta(t).
\end{aligned}$$

Cases 3 and 4 both yield $\|z_1(t)\| \le \alpha\zeta(t)$ on $[a, b]$ with $\zeta(t) = \Theta((t-a)^2)$ for $t \to a$. Moreover, since $m_1 \ne 0$ we can assume that $0 \ne (E_{\mathrm{tk}} \;\; 0)^* = L_{21}$. (Otherwise G consists of a single isolated tank and the DAE reduces to the ODE $A_{\mathrm{tk}}(H_{\mathrm{tk}})H_{\mathrm{tk}}' = 0$, with index 0.) By choosing $\gamma < \epsilon / \left[\delta_{\mathrm{tk}}^{-1}\|L_{12}\|\,\|L_{21}\|\zeta(b)\right]$, we thus ensure (via α) that $\|\delta\|_{C^0} < \epsilon$:

$$\|\delta\|_{C^0([a,b],\mathbb{R}^m)} = \|\delta_2\|_{C^0} \le \|L_{21}\|\,\|z_1\|_{C^0} \le \|L_{21}\|\alpha\zeta(b) < \epsilon.$$

Since $\zeta(t) = \Theta((t-a)^2)$, there exists $t_* > a$ such that $\zeta(t_*)/(t_* - a) < \gamma/(c\alpha\|L_{21}\|)$, which implies

$$\|z(t_*)\| = \|z_2(t_*)\| = (t_* - a)\gamma > c\alpha\|L_{21}\|\zeta(t_*) \ge c\left(0 + \|\delta\|_{C^0([a,t_*],\mathbb{R}^m)}\right).$$

This completes the proof. □

To derive criteria for the index-1 case, i.e., for nonsingularity of the matrix $W_{22}(Q_{\mathrm{pi}})$, let us define the following submatrices:

$$\begin{aligned}
D(Q_{\mathrm{pi}}) &:= \mathrm{Diag}(\varphi_{\mathrm{pi}}'(Q_{\mathrm{pi}}), -\Pi_{\mathrm{pu}}^{Q}, +\Pi_{\mathrm{vl}}^{Q}) \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{A}|}, \\
\tilde{E}_{\mathrm{jc},0} &:= \begin{pmatrix} E_{\mathrm{jc,pi}} & E_{\mathrm{jc,pu}}\Pi_{\mathrm{pu}}^{H} & E_{\mathrm{jc,vl}}\Pi_{\mathrm{vl}}^{H} \end{pmatrix} \in \mathbb{R}^{|\mathcal{N}_{\mathrm{jc}}|\times|\mathcal{A}|}, \\
E_{\mathrm{jc},0} &:= \begin{pmatrix} E_{\mathrm{jc,pu}}(\Sigma_{\mathrm{pu}}^{H})^* & E_{\mathrm{jc,vl}}(\Sigma_{\mathrm{vl}}^{H})^* \end{pmatrix} \in \mathbb{R}^{|\mathcal{N}_{\mathrm{jc}}|\times|\mathcal{A}_{\mathrm{pu}}^{H}\cup\mathcal{A}_{\mathrm{vl}}^{H}|}.
\end{aligned}$$

Note that $E_{\mathrm{jc},0}$ is obtained from $\tilde{E}_{\mathrm{jc},0}$ by removing $E_{\mathrm{jc,pi}}$ and the zero columns associated with flow-controlled arcs.

Proposition 5. *The inverse $W_{22}(Q_{\mathrm{pi}})^{-1}$ exists if and only if*

(a) $\tilde{E}_{\mathrm{jc},0}$ *has full row rank, i.e.,* $\mathrm{rank}(\tilde{E}_{\mathrm{jc},0}) = |\mathcal{N}_{\mathrm{jc}}|$*;*

(b) $E_{\mathrm{jc},0}$ *has full column rank, i.e.,* $\mathrm{rank}(E_{\mathrm{jc},0}) = |\mathcal{A}_{\mathrm{pu}}^{H} \cup \mathcal{A}_{\mathrm{vl}}^{H}|$.

Proof. From the structure of $W(Q_{\mathrm{pi}})$ it is apparent that

$$W_{22}(Q_{\mathrm{pi}}) = \begin{pmatrix} D(Q_{\mathrm{pi}}) & E_{\mathrm{rs}}^* & E_{\mathrm{jc}}^* \\ 0 & I & \\ \tilde{E}_{\mathrm{jc},0} & & 0 \end{pmatrix}.$$

Due to the unit diagonal block associated with the reservoirs, nonsingularity of W_{22} is equivalent to nonsingularity of the submatrix

$$\begin{pmatrix} D(Q_{\mathrm{pi}}) & E_{\mathrm{jc}}^* \\ \tilde{E}_{\mathrm{jc},0} & 0 \end{pmatrix}. \tag{10}$$

The order of this matrix is $|\mathcal{A}| + |\mathcal{N}_{jc}|$ where $|\mathcal{A}| \geq |\mathcal{N}_{jc}|$ by the topological assumptions (T1), (T2) and Proposition 1. Since $\tilde{E}_{jc,0}$ is obtained from E_{jc} by zeroing out some columns (the ones associated with flow-controlled arcs), standard saddle point theory [2] thus implies that (10) is nonsingular iff:

(a) $\tilde{E}_{jc,0}$ has full row rank (implying full column rank of E_{jc}^*);

(b′) $D(Q_{pi})$ is nonsingular on the null space $\ker(\tilde{E}_{jc,0})$.

Now, by definition of the projection matrices $\Pi_{pu}^Q, \Pi_{vl}^Q, \Pi_{pu}^H, \Pi_{vl}^H$ and since $\varphi_{pi}'(Q_{pi}) > 0$, condition (b′) is equivalent to (b): full column rank of $E_{jc,0}$. $\square$

Remark 4. The proof confirms Proposition 3 in that nonsingularity of $W_{22}(Q_{pi})$ is a structural property: $W_{22}(Q_{pi})^{-1}$ either exists everywhere on $\mathbb{R}^{|\mathcal{A}_{pi}|}$ or nowhere.

Remark 5. Item (a) in the proof makes use of the fact that the relevant submatrix (10) has a near-perfect KKT structure. A more detailed derivation would use the unit diagonal entries of $-\Pi_{pu}^Q, +\Pi_{vl}^Q$ in $D(Q_{pi})$ to eliminate "flow-controlled" rows and columns, showing that nonsingularity of (10) is equivalent to nonsingularity of the perfect KKT submatrix

$$\begin{pmatrix} \varphi_{pi}'(Q_{pi}) & & & E_{jc,pi}^* \\ & 0 & & \Sigma_{pu}^H E_{jc,pu}^* \\ & & 0 & \Sigma_{vl}^H E_{jc,vl}^* \\ E_{jc,pi} & E_{jc,pu}(\Sigma_{pu}^H)^* & E_{jc,vl}(\Sigma_{vl}^H)^* & 0 \end{pmatrix} = \begin{pmatrix} \varphi_{pi}'(Q_{pi}) & & E_{jc,pi}^* \\ & 0 & E_{jc,0}^* \\ E_{jc,pi} & E_{jc,0} & 0 \end{pmatrix}.$$

In this form, the invertibility conditions (a), (b) become even more apparent.

Having derived algebraic conditions for nonsingularity of $W(Q_{pi})$, we are now ready to state purely topological index criteria for the DAE. To this end, consider the network subgraph G_0 induced by the pressure-controlled arcs $\mathcal{A}_{pu}^H \cup \mathcal{A}_{vl}^H$ (columns of $E_{jc,0}$), and the larger subgraph $\tilde{G}_0$ induced by $\mathcal{A}_{pi} \cup \mathcal{A}_{pu}^H \cup \mathcal{A}_{vl}^H$ (nonzero columns of $\tilde{E}_{jc,0}$):

$$\begin{aligned} G_0 &= G(\mathcal{A}_{pu}^H \cup \mathcal{A}_{vl}^H), \\ \tilde{G}_0 &= G(\mathcal{A}_{pi} \cup \mathcal{A}_{pu}^H \cup \mathcal{A}_{vl}^H). \end{aligned}$$

Theorem 3. *Let* G_0 *consist of connected components* $G_1, \dots, G_n$*, and let* $\tilde{G}_0$ *consist of components* $\tilde{G}_1, \dots, \tilde{G}_{\tilde{n}}$*. Then the DAE* (6) *is index-1 if and only if*

(a)* $\tilde{G}_0$ *contains* $\mathcal{N}_{jc}$ *and satisfies* $|\tilde{G}_\nu \cap (\mathcal{N}_{rs} \cup \mathcal{N}_{tk})| \geq 1$ *for* $\nu = 1, \dots, \tilde{n}$*;*

(b)* G_0 *is acyclic and satisfies* $|G_\nu \cap (\mathcal{N}_{rs} \cup \mathcal{N}_{tk})| \leq 1$ *for* $\nu = 1, \dots, n$*.*

Proof. Using Proposition 1, we show that (a*) $\Leftrightarrow$ (a) and (b*) $\Leftrightarrow$ (b).

(b) $\Rightarrow$ (b*): Let $\bar{E}_{jc,0}$ denote the nonzero rows of $E_{jc,0}$, determining the column rank. These rows correspond precisely to the junctions in G_0, and thus to a row subset of $E(G_0)$. Now $E(G_0)$ has full column rank iff G_0 is acyclic, which is thus necessary for full column rank of $E_{jc,0}$. Observe next that $\bar{E}_{jc,0}$ admits a decomposition $\bar{E}_{jc,0} = \mathrm{Diag}(\{\bar{E}_{jc,\nu}\})$ induced by the components G_ν. If some G_ν contains two reservoirs or tanks, then we can connect them by a path $P \subseteq G_\nu$ of length $l \geq 1$, traversing $l-1$ junctions. Thus $E(P) \in \mathbb{R}^{(l+1)\times l}$ has rank l whereas the common $(l-1, l)$-submatrix of $E(P)$ and $\bar{E}_{jc,\nu}$ has only rank $l-1$. Since $\bar{E}_{jc,\nu}$ and $E_{jc,\nu}$ have no further entries in these l columns, $E_{jc,0}$ cannot have full column rank.

(b*) $\Rightarrow$ (b): If $|G_\nu \cap (\mathcal{N}_{rs} \cup \mathcal{N}_{tk})| \leq 1$ for all ν, then each $E_{jc,\nu}$ is obtained from $E(G_\nu)$ by deleting at most one row. Thus

$$\mathrm{rank}(E_{jc,\nu}) = \mathrm{rank}(E(G_\nu)).$$

By virtue of the block decompositions of $E_{jc,0}$ and $E(G_0)$, this implies

$$\mathrm{rank}(E_{jc,0}) = \sum_{\nu=1}^{n} \mathrm{rank}(E_{jc,\nu}) = \sum_{\nu=1}^{n} \mathrm{rank}(E(G_\nu)) = \mathrm{rank}(E(G_0)).$$

If, in addition, G_0 is acyclic, we finally obtain $\mathrm{rank}(E_{jc,0}) = \mathrm{rank}(E(G_0)) = |\mathcal{A}_{pu}^H \cup \mathcal{A}_{vl}^H|$.

(a) $\Rightarrow$ (a*): Full row rank of $\tilde{E}_{jc,0}$ clearly requires that $\tilde{G}_0$ contains $\mathcal{N}_{jc}$ (otherwise $\tilde{E}_{jc,0}$ has a zero row) and that every component $\tilde{G}_\nu$ contains a non-junction node (otherwise the subset of rows associated with the junctions of $\tilde{G}_\nu$ will be rank-deficient).

(a*) $\Rightarrow$ (a): The stated conditions are also sufficient, since the components $\tilde{G}_\nu$ induce a decomposition $\tilde{E}_{jc,0} = \mathrm{Diag}(\{\tilde{E}_{jc,\nu}\})$ where each $\tilde{E}_{jc,\nu}$, comprising a proper row subset of $E(\tilde{G}_\nu)$, has full row rank. This completes the proof. □

Remark 6. Condition (b*) has a simple physical interpretation: the pressure differences around each cycle sum up to zero, and the total pressure difference along a path between two reservoirs and/or tanks is uniquely determined by $\bar{H}_{rs}$ (reservoirs) or continuity of the differential variables H_{tk} (tanks). However, the pressure difference along a cycle or path consisting exclusively of pressure-controlled pumps and/or valves is given by the control input, and hence will not match the proper value in general.

Corollary 3. *Under conditions (a*), (b*) of Theorem* 3 *the following hold.*

(a†) G *contains no junction to which only flow-controlled arcs are incident.*
(b†) If all pumps and valves are flow-controlled, $\mathcal{A}^H_{pu} \cup \mathcal{A}^H_{vl} = \emptyset$*, then the DAE is index-1 iff* $\tilde{G}_0 \equiv G(\mathcal{A}_{pi})$ *contains* $\mathcal{N}_{jc}$ *and satisfies* $|\tilde{G}_\nu \cap (\mathcal{N}_{rs} \cup \mathcal{N}_{tk})| \geq 1$ *for* $\nu = 1, \dots, \tilde{n}$*. In specific, at least one reservoir or tank is connected to a pipe.*
(c†) If all pumps and valves are pressure-controlled, $\mathcal{A}^Q_{pu} \cup \mathcal{A}^Q_{vl} = \emptyset$*, then the DAE is index-1 iff* G_0 *is acyclic and satisfies* $|G_\nu \cap (\mathcal{N}_{rs} \cup \mathcal{N}_{tk})| \leq 1$ *for* $\nu = 1, \dots, n$*.*

Proof. Item (a†) follows immediately from (a*). If $\mathcal{A}^H_{pu} \cup \mathcal{A}^H_{vl} = \emptyset$, then (b*) is trivially satisfied, implying (b†). If, finally, $\mathcal{A}^Q_{pu} \cup \mathcal{A}^Q_{vl} = \emptyset$, then (a*) is trivially satisfied due to assumptions (T1) and (T2): $G = \tilde{G}_0 = \tilde{G}_1$ ($\tilde{n} = 1$), yielding (c†). □

Remark 7. Statement (a†) also has a simple physical interpretation: if all arcs incident to a junction are flow-controlled, then the flow balance condition at that junction will be violated in general. Finally, the special cases addressed in (b†) and (c†) turn out to be uncommon in practice: reservoirs and tanks are typically connected to pumps or valves (in contrast to b†), and groundwater reservoirs in specific are often connected to tanks via pumps (in contrast to c†).

To generalize the results of Theorem 3, let us finally extend the model by allowing lossless pipes $\mathcal{A}^\circ_{pi} \subseteq \mathcal{A}_{pi}$ satisfying $\varphi_{ij}(Q_{ij}) \equiv 0$. Clearly, $\varphi'_{pi}(Q_{pi})$ is now only positive semidefinite since each lossless pipe introduces a zero diagonal entry—and hence a potential rank deficiency of $W_{22}(Q_{pi})$. Recall that $\tilde{G}_0$ denotes the subgraph of G induced by all pipes and pressure-controlled arcs (nonzero columns of $\tilde{E}_{jc,0}$), and replace G_0 with G°_0, the subgraph of G induced by lossless pipes and pressure-controlled arcs,

$$\tilde{G}_0 := G(\mathcal{A}_{pi} \cup \mathcal{A}^H_{pu} \cup \mathcal{A}^H_{vl}),$$
$$G^\circ_0 := G(\mathcal{A}^\circ_{pi} \cup \mathcal{A}^H_{pu} \cup \mathcal{A}^H_{vl}).$$

The submatrix of W_{22} associated with G°_0 is

$$E^\circ_{jc,0} := \begin{pmatrix} E^\circ_{jc,pi} & E_{jc,pu}(\Sigma^H_{pu})^* & E_{jc,vl}(\Sigma^H_{vl})^* \end{pmatrix}.$$

Theorem 4 (Main result)**.** *Let* G°_0 *consist of connected components* $G^\circ_1, \dots, G^\circ_{n^\circ}$*, and let* $\tilde{G}_0$ *consist of components* $\tilde{G}_1, \dots, \tilde{G}_{\tilde{n}}$*. Then the DAE* (6) *is index-1 if and only if*

(a‡) $\tilde{G}_0$ *contains* $\mathcal{N}_{jc}$ *and satisfies* $|\tilde{G}_\nu \cap (\mathcal{N}_{rs} \cup \mathcal{N}_{tk})| \geq 1$ *for* $\nu = 1, \dots, \tilde{n}$*;*
(b‡) G°_0 *is acyclic and satisfies* $|G^\circ_\nu \cap (\mathcal{N}_{rs} \cup \mathcal{N}_{tk})| \leq 1$ *for* $\nu = 1, \dots, n^\circ$*.*

Proof. This is proved exactly as Theorem 3, except that G_0 is replaced with the larger subgraph G°_0, the submatrix $E_{jc,0}$ is replaced with $E^\circ_{jc,0}$, and full column rank amounts to $\mathrm{rank}(E^\circ_{jc,0}) = |\mathcal{A}^\circ_{pi} \cup \mathcal{A}^H_{pu} \cup \mathcal{A}^H_{vl}|$. □

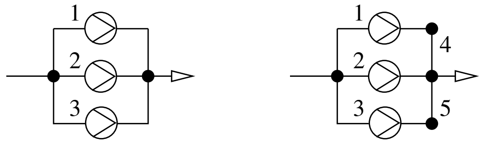

FIGURE 1. Parallel pumps in multigraph and graph representation

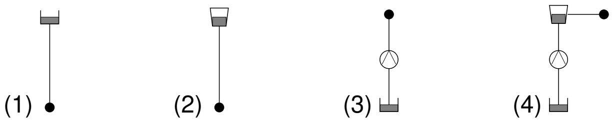

FIGURE 2. Simple networks analyzed in examples (1)–(4).

Remark 8. Lossless pipes are primarily useful to convert multigraphs (with parallel arcs) into simple graphs by introducing fictitious zero-length pipes. Groups of pumps operated in parallel present a typical situation of this kind; see Fig. 1. Condition (b‡) here implies that at most one pump in the group may be pressure-controlled; all others must be flow-controlled to avoid cycles of pressure-controlled arcs.

Another situation where lossless pipes might seem useful is ill-conditioning of $\varphi'_{\mathrm{pi}}(Q_{\mathrm{pi}})$ caused by very short pipes with negligible pressure loss. Such ill-conditioning should generally *not* be prevented by declaring the short pipes lossless, however: it is typically possible (and better) to remove them entirely from the model.

Corollary 4. *If all pipes are lossless, $\mathcal{A}^\circ_{\mathrm{pi}} = \mathcal{A}_{\mathrm{pi}}$, then the DAE* (6) *is index-1 if and only if G°_0 is acyclic, contains $\mathcal{N}_{\mathrm{jc}}$, and every component G°_ν contains precisely one reservoir or tank, $|G^\circ_\nu \cap (\mathcal{N}_{\mathrm{rs}} \cup \mathcal{N}_{\mathrm{tk}})| = 1$ for $\nu = 1, \dots, n^\circ$.*

Proof. This is immediate from Theorem 4 since $G^\circ_0 = \tilde{G}_0$. □

Remark 9. The corollary applies to networks of small spatial dimensions where all pipe friction losses are negligible in absolute terms. However, unless the very special conditions on the network subgraph G°_0 are satisfied, relative losses do matter, and simply neglecting them will introduce artificial singularities.

4. EXAMPLES

To illustrate the theoretical results, we analyze the four sample networks in Fig. 2.

Example 1. Consider first the most trivial water network possible, consisting of a reservoir, a pipe, and a junction (customer). By Theorem 4, the associated "DAE" is always index-1, regardless of whether the pipe is lossless or not: we have $\tilde{G}_0 = G(\mathcal{A}_{\mathrm{pi}}) = G$ (which contains $\mathcal{N}_{\mathrm{jc}}$ and a reservoir) and $G^\circ_0 = G(\mathcal{A}^\circ_{\mathrm{pi}}) \in \{\emptyset, G\}$ (which is acyclic and contains no more than one tank or reservoir). This result is also immediate from the equations. With $x = x_2 = (Q_{\mathrm{pi}}, H_{\mathrm{rs}}, H_{\mathrm{jc}})$, the hydraulic model (3) and its invertible matrix W_{22} read

$$F(x', x, t) = \begin{pmatrix} \varphi_{\mathrm{pi}}(Q_{\mathrm{pi}}) - H_{\mathrm{rs}} + H_{\mathrm{jc}} \\ H_{\mathrm{rs}} - \bar{H}_{\mathrm{rs}} \\ Q_{\mathrm{pi}} - D_{\mathrm{jc}} \end{pmatrix}, \quad W_{22}(Q_{\mathrm{pi}}) = \left(\begin{array}{c|cc} \varphi'_{\mathrm{pi}}(Q_{\mathrm{pi}}) & -1 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right),$$

yielding the unique C^0-solution $x = (D_{\mathrm{jc}}, \bar{H}_{\mathrm{rs}}, \bar{H}_{\mathrm{rs}} - \varphi_{\mathrm{pi}}(D_{\mathrm{jc}}))$.

Example 2. In the second example we replace the reservoir with a tank. Again we have $\tilde{G}_0 = G(\mathcal{A}_{\mathrm{pi}}) = G$ (containing $\mathcal{N}_{\mathrm{jc}}$ and a tank) and $G^\circ_0 = G(\mathcal{A}^\circ_{\mathrm{pi}}) \in \{\emptyset, G\}$ (being acyclic and containing just one tank), so that the DAE is always index-1. Letting $x_1 = H_{\mathrm{tk}}$ and

$x_2 = (Q_{pi}, H_{jc})$, the model (3) and invertible matrix W_{22} now read

$$F(x', x, t) = \begin{pmatrix} -A_{tk}(H_{tk})H'_{tk} - Q_{pi} \\ -H_{tk} + \varphi_{pi}(Q_{pi}) + H_{jc} \\ Q_{pi} - D_{jc} \end{pmatrix}, \qquad W_{22}(Q_{pi}) = \left(\begin{array}{c|c} \varphi'_{pi}(Q_{pi}) & 1 \\ \hline 1 & 0 \end{array}\right).$$

We obtain a set of unique C^1_P-solutions parameterized by $H_{tk}(t_0)$, $t_0 \in \bar{\mathcal{I}}$ (the initial value of the ODE): $x = (H_{tk}, D_{jc}, H_{tk} - \varphi_{pi}(D_{jc}))$ where

$$H_{tk}(t) = H_{tk}(t_0) - \int_{t_0}^{t} A_{tk}(H_{tk}(s))^{-1} D_{jc}(s)\, ds.$$

Example 3. In the next example, the water is pumped from the reservoir to the customer.

(a) If the pump is pressure-controlled, we have $\tilde{G}_0 = G^\circ_0 = G(\mathcal{A}_{pu}) = G$, which is acyclic and contains $\mathcal{N}_{jc}$ and precisely one reservoir. Thus the model is again index-1 by Theorem 4. With $x = x_2 = (Q_{pu}, H_{rs}, H_{jc})$ and $u = \Delta H_{pu}$, the DAE (6) now becomes linear with constant coefficients (in fact, a family of linear equation systems over $\mathcal{I}$),

$$F(x', x, t) = Ax' + Bx - r(t),$$

where

$$A = 0, \qquad B = W_{22} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad r = \begin{pmatrix} \Delta H_{pu} \\ \bar{H}_{rs} \\ D_{jc} \end{pmatrix}.$$

The unique C^0-solution is $x = (D_{jc}, \bar{H}_{rs}, \bar{H}_{rs} + \Delta H_{pu})$. Of course, linear DAE theory (see, e.g., [3, §2.3]) yields the same results here: since $\det(B) \neq 0$, the matrix pencil $\{\lambda A + B : \lambda \in \mathbb{C}\} \equiv \{B\}$ is regular, which implies solvability. Further, the DAE is index-1 since A has nilpotency one ($A^1 = 0$ but $A^0 \neq 0$).

(b) If the pump is flow-controlled, we have $\tilde{G}_0 = G^\circ_0 = \emptyset$, and hence the DAE is *not* index-1. In fact, it is not even solvable in this case: with $x = x_2 = (\Delta H_{pu}, H_{rs}, H_{jc})$ and $u = Q_{pu}$, the DAE (6) again becomes linear with constant coefficients,

$$A = 0, \qquad B = W_{22} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ \bar{H}_{rs} \\ D_{jc} - Q_{pu} \end{pmatrix}.$$

Here, since $\det(B) = 0$, the matrix pencil $\{\lambda A + B\}$ is not regular. Therefore the DAE is not solvable and the index is undefined. From the structure of B and r it is apparent that H_{jc} and ΔH_{pu} are undetermined (only the difference $H_{jc} - \Delta H_{pu}$ is uniquely determined), and that the third equation is inconsistent unless $Q_{pu} \equiv D_{jc}$ (in which case it is superfluous). This reflects the obvious fact that one simply cannot control the pump flow Q_{pu} whenever D_{jc} is given.

Example 4. In the final example we insert a tank and a pipe between pump and customer.

(a) If the pump is flow-controlled, we have $\tilde{G}_0 = G(\mathcal{A}_{pi})$ (containing $\mathcal{N}_{jc}$ and the tank) and $G^\circ_0 = G(\mathcal{A}^\circ_{pi}) \in \{\emptyset, \tilde{G}_0\}$ (being acyclic and containing at most one tank); the model is again index-1. Letting $x_1 = H_{tk}$, $x_2 = (Q_{pi}, \Delta H_{pu}, H_{rs}, H_{jc})$, and $u = Q_{pu}$, the DAE (6) and invertible matrix W_{22} read

$$F = \begin{pmatrix} -A_{tk}(H_{tk})H'_{tk} - Q_{pi} + Q_{pu} \\ -H_{tk} + \varphi_{pi}(Q_{pi}) + H_{jc} \\ H_{tk} - \Delta H_{pu} - H_{rs} \\ H_{rs} - \bar{H}_{rs} \\ Q_{pi} - D_{jc} \end{pmatrix}, \qquad W_{22} = \left(\begin{array}{cc|cc} \varphi'_{pi}(Q_{pi}) & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right),$$

yielding the unique C^1_P-solutions $x = (H_{tk}, D_{jc}, H_{tk} - \bar{H}_{rs}, \bar{H}_{rs}, H_{tk} - \varphi_{pi}(D_{jc}))$ where

$$H_{tk}(t) = H_{tk}(t_0) + \int_{t_0}^{t} A_{tk}(H_{tk}(s))^{-1} \left[Q_{pu}(s) - D_{jc}(s)\right] ds.$$

(b) If the pump is pressure-controlled, we have $\tilde{G}_0 = G(\mathcal{A}_{pi} \cup \mathcal{A}_{pu}) = G$ (containing $\mathcal{N}_{jc}$, a reservoir, and a tank) and $G_0^\circ = G(\mathcal{A}_{pi}^\circ \cup \mathcal{A}_{pu}) \in \{G(\mathcal{A}_{pu}), G\}$ (being acyclic but containing the reservoir *and* the tank in any case); thus the model is *not* index-1. Here we encounter a more subtle type of singularity. Letting $x_1 = H_{tk}$, $x_2 = (Q_{pi}, Q_{pu}, H_{rs}, H_{jc})$, and $u = \Delta H_{pu}$, the DAE (6) and matrix W_{22} read

$$F = \begin{pmatrix} -A_{tk}(H_{tk})H_{tk}' - Q_{pi} + Q_{pu} \\ -H_{tk} + \varphi_{pi}(Q_{pi}) + H_{jc} \\ H_{tk} - H_{rs} - \Delta H_{pu} \\ H_{rs} - \bar{H}_{rs} \\ Q_{pi} - D_{jc} \end{pmatrix}, \qquad W_{22} = \left(\begin{array}{cc|cc} \varphi_{pi}'(Q_{pi}) & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right).$$

Of course, F has the same form as in the flow-controlled case, but W_{22} becomes singular since only the pressure difference ΔH_{pu} appears in the pump equation but not the flow Q_{pu}. A (unique) solution still exists, $x = (H_{tk}, D_{jc}, A_{tk}(H_{tk})H_{tk}' + D_{jc}, \bar{H}_{rs}, H_{tk} - \varphi_{pi}(D_{jc}))$, where $H_{tk} = \bar{H}_{rs} + \Delta H_{pu}$. However, for this to be a C_P^1-solution one must require that $\bar{H}_{rs} + \Delta H_{pu} \in C^1(\bar{\mathcal{I}}, \mathbb{R})$: the control ΔH_{pu} is restricted to an affine subspace of $C^0(\bar{\mathcal{I}}, \mathbb{R})$. Moreover, the solution here is not parameterized by $H_{tk}(t_0)$. A closer inspection reveals that the DAE in this case has differentiation index 2.

5. Conclusions

The results of this article show that our hydraulic DAE model possesses highly desirable properties under weak assumptions on data regularity and network topology: the DAE is index-1 by several criteria and on the entire domain of definition. Although we have studied the model under weak regularity assumptions, even weaker requirements are appropriate in application models [4]: the data functions are generally only piecewise continuous with respect to an equidistant time grid Γ. This means that our analysis applies only to the subintervals of Γ, and that jumps in the algebraic variable x_2 and in the derivative x_1' may occur at the grid points. There are several sources of discontinuities:

(a) The predicted demand D_{jc} is typically modeled as a piecewise constant function.
(b) Pumps and valves are usually operated with piecewise constant control, Q_j or ΔH_j.
(c) Conceptual tanks $j \in \mathcal{N}_{tk}$ may consist of several physical tanks $j\nu$, $\nu = 1, \dots, N_j$, each of which may be temporarily unavailable. The effective tank areas thus become discontinuous in time, $A_j(H_j, t) = \sum_\nu Y_{j\nu}(t)A_{j\nu}(H_j)$ with $Y_{j\nu}(t) \in \{0, 1\}$.
(d) Pumps and valves may be pressure-controlled during certain periods of time and flow-controlled during other periods. The sets $\mathcal{A}_{pu}^Q$, $\mathcal{A}_{vl}^Q$, $\mathcal{A}_{pu}^H$, $\mathcal{A}_{vl}^H$ and the entry positions of W_{22} are only piecewise constant in this case, inducing structural changes in the model.

All these issues also occur in operative planning for real-world water networks [4, 5]. Planning models of this type are naturally formulated as DAE boundary value problems (BVP), and the discontinuity issues (a)–(d) are conveniently handled by a full discretization approach ("discretize-then-optimize") using the given time grid Γ, or a refinement. However, there are some significant differences in comparison to DAE simulation.

First, in the BVP context there is no need to select control variables: they are dynamic variables just like the states. Consequently, the full Wronskian W (in a discretized version) becomes more relevant than W_{22}. A BVP solver may actually determine possible control variables on the fly by suitable pivoting on W. Moreover, in a full discretization approach the entire planning horizon is treated simultaneously, involving a different copy of W (and of further linearized constraints) in each time step. In such a context, the theoretical results of this paper provide a basis for constructing sophisticated solution algorithms that employ structural pivoting strategies largely based on the network topology rather than the data of the current iterate.

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