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Optimum Path Packing on Wheels: The Consecutive Case

Preprint SC-95-31 (November 1995)

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Abstract

We show that, given a wheel with nonnegative edge lengths and pairs of terminals located on the wheel's outer cycle such that the terminal pairs are in consecutive order, then a path packing, i. e., a collection of edge disjoint paths connecting the given terminal pairs, of minimum length can be found in strongly polynomial time. Moreover, we exhibit for this case a system of linear inequalities that provides a complete and nonredundant description of the path packing polytope, which is the convex hull of all incidence vectors of path packings and their supersets.

1 Introduction

The topic of packing paths, trees, Steiner trees etc. into graphs has received considerable and strongly growing attention in the recent fifteen years. Two sources nourish the development; one is the increasing demand from VLSI design for routing algorithms, the other is the discovery of beautiful results such as the Okamura-Seymour theorem [7] that provide new insights and are the basis of many modifications and generalizations. Excellent surveys of these developments can be found, for instance, in [2] and [8].

Most of these results are of the following type. Given a graph (with some additional properties) and a collection of sets of terminals, then a packing of paths (or trees or Steiner trees etc.) exists provided that some conditions (typically conditions on certain cuts in the graph) hold. Frequently, the proofs yield polynomial time algorithms for finding such a packing. Unfortunately, the graph properties needed for the existence of such results are very restrictive and only occasionally helpful for solving problems in VLSI design. Questions of this type are $\mathcal{NP}$-hard not only in general but even for classes of graphs that appear rather special.

VLSI designers are usually happy to find some routing of the given terminal sets; however, they would be much more interested in determining routings that are minimal with respect to certain criteria such as the total wire length. This problem turns out to be $\mathcal{NP}$-hard for basically all practically relevant cases.

Nevertheless, currently the first steps are being made to attack the optimum packing problem by means of branch and cut algorithms (and the like) that have the potential to produce optimum or provably good solutions, see [4], [5]. To our knowledge, there are only very few special cases known for which optimum packing problems can be solved in polynomial time (see, for instance, [3]). We present another such case here. We show that if a wheel with nonnegative edge lengths is given and if the terminal pairs are consecutively located on the wheel's outer cycle, then a list of pairwise edge disjoint paths connecting the terminals pairs (short: a path packing) that has minimum total length can be found in polynomial time. Moreover, we are able to give a complete linear description of the path packing polytope, i e., the convex hull of all incidence vectors of path packings and supersets of path packings. This seems to be the first result of this type.

The polyhedral description of the path packing polytope in this case requires technical effort and is rather surprising. If there is an even number of terminal pairs polynomially many inequalities suffice, while for an odd number of terminal pairs, exponentially many inequalities are needed.

2 A Polynomial Time Algorithm

In this chapter we present a polynomial time algorithm that solves the problem of packing edge disjoint paths on a wheel, provided that the terminals l_i, r_i are consecutively located on the outer cycle of the wheel ($i = 1, \ldots, k$). Before explaining the algorithm let us introduce some notation that we use throughout this paper.

We denote a graph by $G = (V, E)$, where V is the node set and E the edge set. Our graphs do neither have parallel edges nor loops. An edge e with endnodes u and v is denoted by uv or, if possible ambiguity requires it, by $[u, v]$. Let $K = (v_0, e_1, v_1, e_2, \ldots, v_{l-1}, e_l, v_l)$ be a sequence of nodes and edges, where each edge e_i is incident with the nodes v_{i-1} and v_i for $i = 1, \ldots, l$, and where the edges are pairwise different and the nodes distinct (except possibly v_0 and v_l). K is called a *path* (or a $[v_0, v_l]-$*path*), if $v_0 \neq v_l$, and a *cycle*, if $v_0 = v_l$ and $l \geq 2$. For our purposes, it is appropriate to consider a path P or a cycle C, respectively, as a subset of the edge set.

A *wheel* consists of a cycle and a center connected to all nodes of the cycle by an edge, more formally: a *wheel with n spokes and center z* is a graph $G = (V, E)$ consisting of n nodes numbered $\{1, \ldots, n\}$ and a special node z, i. e., $V := \{1, \ldots, n\} \cup \{z\}$, and an edge set $E := C \cup S$ with $C := \{[i, i+1] \mid i = 1, \ldots, n\}$ and $S := \{[z, i] \mid i = 1, \ldots, n\}$. The edges in S are called *spokes*, and we assume that the nodes of C are numbered in clockwise order around z. (To make index computations notationally easier we identify an index $i > n$ with

$((i-1)$ modulo $n) + 1$.) We call a list of node sets $T_1, \ldots, T_k$, $k \geq 2$ of the outer cycle C *in consecutive order*, if all nodes $l_i, r_i \in T_i$, $l_i < r_i$, $i = 1, \ldots, k$, appear in the sequence $l_1, r_1, l_2, r_2, \ldots, l_k, r_k$ by walking along C. We denote the cut $\{uv \in E \mid u \in X, v \notin X\}$ induced by some node set $X \subseteq V$ by the symbol $\delta(X)$. For $c \in \mathbb{R}^E$ and $F \subseteq E$, we define $c(F) := \sum_{e \in F} c_e$.

Finally, to facilitate technical arguments when dealing with a wheel with n spokes and center z, we introduce, for $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, n-1\}$, the following symbols.

Nodes on the interval along C from i to $i+j$:

$$[i : i+j] \quad := \{i + r \mid r = 0, \ldots, j\}.$$

Spokes connecting the interval $[i : i+j]$ to the center:

$$S(i : i+j) \quad := \{[z, i+r] \mid r = 0, \ldots, j\}.$$

Edges of the interval $[i : i+j]$:

$$\begin{aligned} C(i : i+j) \quad &:= \{[r, r+1] \mid r = i, \ldots, i+j-1\}, \quad && \text{if } j > 0, \\ C(i : i+j) \quad &:= \emptyset, && \text{if } j = 0. \end{aligned}$$

Closed fan of the interval $[i : i+j]$, i. e., all edges of the interval and the corresponding spokes:

$$F[i : i+j] \quad := C(i : i+j) \cup S(i : i+j).$$

Open fan of the interval $[i : i+j]$, i. e., closed fan without outer spokes:

$$\begin{aligned} F(i : i+j) \quad &:= C(i : i+j) \cup S(i+1 : i+j-1), \quad && \text{if } j \geq 2, \\ F(i : i+j) \quad &:= C(i : i+j), && \text{if } j = 1, \\ F(i : i+j) \quad &:= \emptyset, && \text{if } j = 0. \end{aligned}$$

Right open fan of the interval $[i : i+j]$, i. e., closed fan without right outer spoke:

$$\begin{aligned} F[i : i+j) \quad &:= C(i : i+j) \cup S(i : i+j-1), \quad && \text{if } j > 0, \\ F[i : i+j) \quad &:= \emptyset, && \text{if } j = 0. \end{aligned}$$

Using this notation, our path packing problem can be formulated as follows.

Problem 2.1 (Packing paths with consecutive sets of terminals on a wheel)

Instance:

A wheel $G = (V, E)$ with nonnegative edge lengths $w_e \in \mathbb{R}$, $e \in E$.

A number $k \in \mathbb{N}$ and a list of node pairs $\mathcal{T} = \{\{l_1, r_1\}, \ldots, \{l_k, r_k\}\}$ with $l_1 < r_1 < l_2 < r_2 < \ldots < l_k < r_k$.

Problem:

Find edge sets $P_1, \ldots, P_k \subseteq E$ such that

(i) P_i contains a path in G from l_i to r_i for $i = 1, \ldots, k$,

(ii) The sets $P_1, \ldots, P_k$ are mutually edge disjoint,

(iii) $\sum_{i=1}^{k} \sum_{e \in P_i} w_e$ *is minimal.*

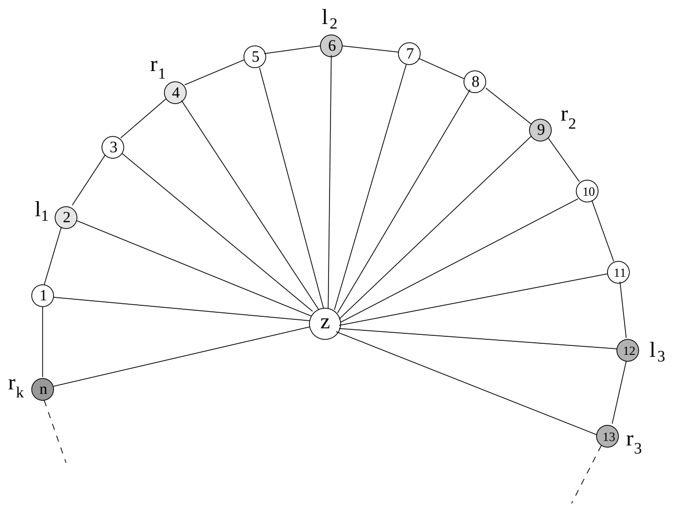

Figure 1:

Each node in $\{l_1, r_1, l_2, r_2, \ldots, l_k, r_k\}$ is called a *terminal*, and each pair of nodes $\{l_i, r_i\}$ $(i = 1, \ldots, k)$ is called a *terminal pair*. We call an edge set P a *packing of paths* or a *path packing* if P can be partitioned into edge sets $P_1, \ldots, P_k$ that satisfy (i) and (ii) of Problem 2.1. A path packing P is called *edge-minimal* if, for every $e \in P$, the set $P \setminus \{e\}$ is not a packing of paths. These definitions slightly deviate from the literature standard since what we term edge-minimal path packing is usually called path packing.

We have the following reasons for this modification. We are interested in VLSI routing, and the length functions coming up in this area are always positive. In this case, every optimum path packing is obviously edge-minimal. Thus, our model can be used to solve problems of this type. Moreover, if negative lengths are allowed then, for general graphs, the shortest path problem is $\mathcal{NP}$-hard and no linear description of the shortest path polytope is known. Therefore, it seems hopeless to investigate edge-minimal path packings with more than one terminal pair. Although it is not difficult to design a polynomial time shortest path algorithm for wheels even if negative lengths are allowed we thought that concentrating on nonnegative length functions and investigating path packings and their supersets seems to be the more promissing and probably further reaching approach.

For arbitrary graphs, the problem of finding an optimal packing of paths is, of course, $\mathcal{NP}$-hard. Even for several special cases, this problem remains $\mathcal{NP}$-hard, e. g., if G is a grid graph ([6]). However, if we restrict G to be a wheel and if we require that the terminal pairs are consecutively located on the outer cycle of G, an optimal packing of paths can be determined in polynomial time.

The idea of this algorithm is based on two observations which we briefly describe now.

It is easy to see that, for every instance of Probem 2.1, there always exists an optimal path packing that is edge-minimal and that has the property that, for every $i \in \{1, \ldots, k\}$, the path that connects the two terminals l_i and r_i uses edges only from the set $F[r_{i-1} : l_{i+1}]$. Hence, such a path from l_i to r_i may only be in "conflict" with such a path from l_{i-1} to r_{i-1} or with such a path from l_{i+1} to r_{i+1}. Further, the number of different paths from l_i to r_i in the subgraph $([r_{i-1} : l_{i+1}] \cup \{z\}, F[r_{i-1} : l_{i+1}])$ of the wheel is polynomial in n.

Let $P_i^1, \ldots, P_i^{s_i}$ denote the different paths from l_i to r_i in the subgraph $([r_{i-1} : l_{i+1}] \cup \{z\}, F[r_{i-1} : l_{i+1}])$. We define a digraph H as follows. With every path P_i^u ($i = 1, \ldots, k,\ u = 1, \ldots, s_i$) we associate a node that we denote by p_i^u. We set $X := \{p_i^u \mid i = 1, \ldots, k,\ u = 1, \ldots, s_i\}$. For every pair p_i^u, p_j^v of nodes in X we introduce the arc (p_i^u, p_j^v) if and only if $j = i + 1$ and the paths P_i^u and P_j^v do not share a common edge. Such an arc receives the length of the path P_i^u. Let Y denote this set of arcs. In the digraph $H = (X, Y)$ we now look for a shortest directed cycle which, as we will see, corresponds to an optimal packing of paths on the given wheel. Consequently, Problem 2.1 can be solved in (strongly) polynomial time.

In the following we discuss this procedure in more detail. We always assume that $G = (V, E)$ is a wheel with nonnegative edge lengths $w_e \in \mathbb{R}$, $e \in E$. Moreover, $\mathcal{T} = \{\{l_1, r_1\}, \ldots, \{l_k, r_k\}\}$, is the list of consecutive terminal pairs and we assume that $l_1 < r_1 < l_2 < r_2 < \ldots < l_k < r_k$.

Note that every edge-minimal path packing P can be partitioned into k edge disjoint paths $P_1, \ldots, P_k$ linking l_i and r_i, $i = 1, \ldots, k$. We call paths $P_1, \ldots, P_k$ with this property a *path partition of* P. Path partitions are not necessarily unique.

Lemma 2.2 *Let P be an edge-minimal packing of paths. Then, P can be partitioned into paths $P_1, \ldots, P_k$ such that for every $i \in \{1, \ldots, k\}$ the following conditions are satisfied.*

(i) $F(l_i, r_i) \cap P_t = \emptyset$ *for all* $t \in \{1, \ldots, k\} \setminus \{i\}$.

(ii) $F[r_i : l_{i+1}] \cap P_t = \emptyset$ *for all* $t \in \{1, \ldots, k\} \setminus \{i, i + 1\}$.

Proof.

We prove (i). We assume that an edge-minimal path packing P exists that cannot be partitioned into k paths satisfying (i). If $P_1, \ldots, P_k$ is any path partition of P, we set $T(P_1, \ldots, P_k) := \{(i, t) \mid i, t \in \{1, \ldots, k\},\ i \neq t \text{ and } F(l_i : r_i) \cap P_t \neq \emptyset\}$. Among all path partitions of P we choose a partition $P_1, \ldots, P_k$ such that

$|T(P_1, \dots, P_k)|$ is minimum. To contradict the assumtion we construct a path partition $P'_1, \dots, P'_k$ with $|T(P'_1, \dots, P'_k)| < |T(P_1, \dots, P_k)|$.

By assumption there are indices $i, t \in \{1, \dots k\}$, $i \neq t$, such that $F(l_i : r_i) \cap P_t \neq \emptyset$. Since P_t does not contain a cycle one of the edges $[l_i, l_i + 1]$ or $[r_i - 1, r_i]$ must belong to P_t, say $[l_i, l_i + 1]$, and moreover, the center z must belong to $V(P_i)$. Let us denote the subpath of P_i linking l_i to z by P_{l_i} and the subpath of P_i linking r_i to z by P_{r_i}, i. e., $P = P_{l_i} \cup P_{r_i}$. Clearly, $P_{l_i} \cap F(l_i : r_i) = \emptyset$. We distinguish the following two cases.

- $[r_i - 1, r_i] \in P_t$: Then obviously $P_{r_i} \cap F(l_i : r_i) = \emptyset$. We set $P'_i := P_t \cap F(l_i, r_i)$ and $P'_t := (P_t \setminus F(l_i : r_i)) \cup P_i$.
- $[r_i - 1, r_i] \notin P_t$: Then $z \in V(P_t)$. Let Q denote the subpath of P_t from l_i to z. We set $P'_i := P_{r_i} \cup Q$ and $P'_t := (P_t \setminus F(l_i : r_i)) \cup P_{l_i}$.

Since P is edge-minimal, in both cases, the edge sets P'_i and P'_t are paths linking l_i to r_i and l_t to r_t, respectively. Setting $P'_j := P_j$, $j = 1, \dots, k$, $i \neq j \neq t$, we have constructed a path partition of P with $|T(P'_1, \dots, P'_k)| < |T(P_1, \dots, P_k)|$ contradicting the minimality assumption. This implies that P must have a path partition satisfying (i).

(ii) follows directly from (i). ∎

Let P be an edge-minimal packing of paths. Due to Lemma 2.2 we know that P can be partitioned into k edge disjoint paths that satisfy the conditions (i) and (ii). Moreover, it is easy to see that these paths are unique. For the remainder of this paper, we denote, for a given edge-minimal packing of paths P, by P_i the (unique) path from l_i to r_i that satisfies $F[l_i : r_i] \cap P_t = \emptyset$ for all $t \in \{1, \dots, k\} \setminus \{i\}$ and $F[r_i : l_{i+1}] \cap P_t = \emptyset$ for all $t \in \{1, \dots, k\} \setminus \{i, i+1\}$. Instead of P we also write $(P_1, \dots, P_k)$.

Lemma 2.3 *For a given $i \in \{1, \dots, k\}$, let $\mathcal{P}_i$ denote the set of edge-minimal paths from l_i to r_i in the subgraph $([r_{i-1} : l_{i+1}] \cup \{z\}, F[r_{i-1} : l_{i+1}])$. The value $|\mathcal{P}_i|$ is bounded by $O(n^2)$.*

Proof. Let $P_i \in \mathcal{P}_i$. If $z \notin V(P_i)$, then P_i is uniquely determined. If $z \in V(P_i)$, there are exactly two edges that are incident to z and that are contained in P_i. Let uz and vz denote these edges. Clearly, $u, v \in V(C)$ and $u \neq v$. W. l. o. g. we may assume $v \in [u : l_{i+1}]$ and hence, the path P_i is the union of the path from l_i to u in the subgraph $([r_{i-1} : l_{i+1}], C(r_{i-1} : l_{i+1}))$ and the path from r_i to v in the subgraph $([r_{i-1} : l_{i+1}], C(r_{i-1} : l_{i+1}))$ and the set $\{uz, vz\}$. Since the number of edges incident to z is bounded by n, the statement follows. ∎

For $i \in \{1, \dots, k\}$, let $P_i^1, \dots, P_i^{s_i}$ denote the different paths from l_i to r_i in the subgraph $([r_{i-1} : l_{i+1}] \cup \{z\}, F[r_{i-1} : l_{i+1}])$. We now define the digraph

$H := (X, Y)$ with arc costs c as follows. With every path P_i^u ($i = 1, \ldots, k$, $u = 1, \ldots, s_i$) we associate a node which we denote by p_i^u. We define X as the corresponding set of nodes. For every pair p_i^u, p_j^v of nodes in X we introduce the arc (p_i^u, p_j^v) if and only if $j = i + 1$ and the paths P_i^u and P_j^v do not share a common edge. We denote this set of arcs by Y. Finally, we define the cost $c(p_i^u, p_{i+1}^v)$ of some arc $(p_i^u, p_{i+1}^v) \in Y$ as the length $w(P_i^u)$ of the path P_i^u.

Figures 2 and 3 illustrate this construction. In Figure 2, a wheel G with the terminal set $\mathcal{T} = \{\{l_1, r_1\}, \{l_2, r_2\}, \{l_3, r_3\}, \{l_4, r_4\}\}$ is shown. For every $1 \leq i \leq 4$, there exist exactly 5 paths $P_i^1, \ldots, P_i^5$ in the subgraph $([r_{i-1} : l_{i+1}], F[r_{i-1} : l_{i+1}])$, namely $P_i^1 = [l_i, r_i]$, $P_i^2 = [l_i, z] \cup [r_i, z]$, $P_i^3 = [l_i, z] \cup [r_i, l_{i+1}] \cup [l_{i+1}, z]$, $P_i^4 = [r_i, z] \cup [r_{i-1}, l_i] \cup [r_{i-1}, z]$ and $P_i^5 = [l_{i+1}, z] \cup [l_{i+1}, r_i] \cup [r_{i-1}, l_i] \cup [r_{i-1}, z]$. Every such path is represented by a node in H as it is shown in Figure 3. An arc (p_i^u, p_{i+1}^v) in H is introduced if the two paths P_i^u and P_{i+1}^v do not intersect.

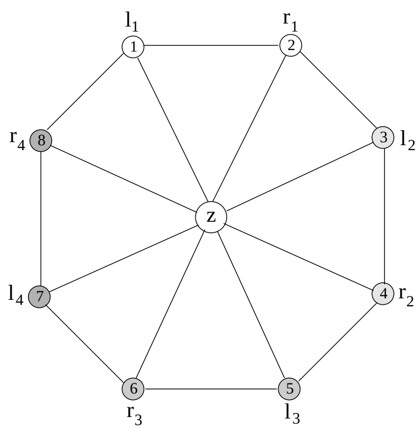

Figure 2:

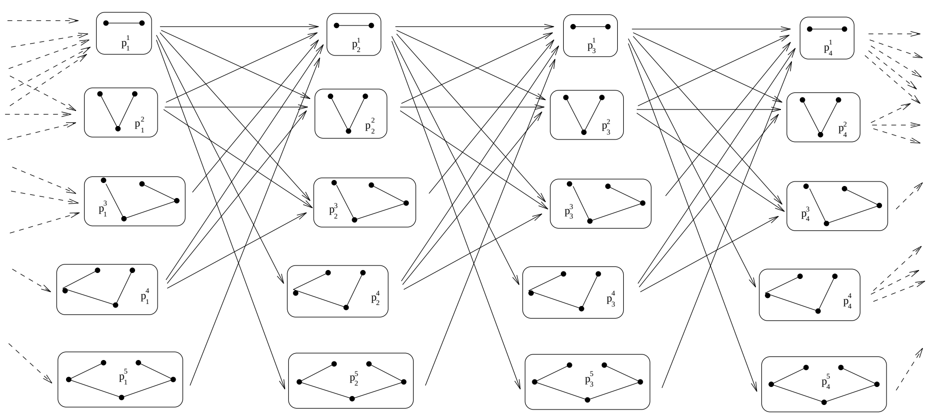

Figure 3:

Due to Lemma 2.3, the size of H is polynomial in n. Moreover, if $P = (P_1, \dots, P_k)$ is a path packing in G, then every such path P_i, $i = 1, \dots, k$, corresponds to a node $p_i^{u_i}$ for some $u_i \in \{1, \dots, s_i\}$. Since P_i and P_j for $i \neq j$ do not share a common edge, the arcs $(p_1^{u_1}, p_2^{u_2}), (p_2^{u_2}, p_3^{u_3}), \dots, (p_k^{u_k}, p_1^{u_1})$ in Y define a directed cycle in H. The cost $c(T)$ of the directed cycle T is equal to the length $w(P)$ of the path packing P by definition. Conversely, every directed cycle $T = \{(p_1^{u_1}, p_2^{u_2}), (p_2^{u_2}, p_3^{u_3}), \dots, (p_k^{u_k}, p_1^{u_1})\}$ in H corresponds to paths $P_i^{u_i}$ from l_i to r_i in the subgraph $([r_{i-1} : l_{i+1}] \cup \{z\}, F[r_{i-1} : l_{i+1}])$ $(i = 1, \dots, k)$. By construction, $P_i^{u_i}$ and $P_j^{u_j}$, $j \neq i$, do not intersect in some edge. Hence, $P := (P_1^{u_1}, \dots, P_k^{u_k})$ is a packing of paths in G and the length $w(P)$ is the same as the cost $c(T)$ of the cycle T.
By applying shortest path or max flow techniques, a directed cycle in H of minimal cost can be computed in time and space complexity that is polynomial in the encoding length of the data. Consequently, an optimal path packing in G can be determined in polynomial time. In fact, strongly polynomial algorithms can be derived. See [1] for a survey of known algorithms of this type.

3 The Path Packing Polytope

Let $W = (V, E)$ be a wheel and let $\mathcal{T} = \{\{l_1, r_1\}, \dots, \{l_k, r_k\}\}$, $l_i, r_i \in V$, $i = 1, \dots, k$ be a list of consecutive terminal pairs. The path packing polytope $\mathrm{PP}\,(W, \mathcal{T})$ is the convex hull of all incidence vectors of path packings P, i.e.,

$$\mathrm{PP}\,(W, \mathcal{T}) := \ \mathrm{conv}\ \{\chi^P \mid P \text{ is a solution of Problem 2.1}\}.$$

Here, $\chi^P \in \mathbb{R}^E$ denotes the incidence vector of the set $P \subseteq E$, i.e., $\chi_e^P := 1$ if $e \in P$ and $\chi_e^P := 0$ if $e \notin P$.

In this section we start the investigation of the path packing polytope $\mathrm{PP}\,(W, \mathcal{T})$. In particular, we introduce the class of *1-cut*, the class of *2-cut* and the class of *windmill* inequalities. We will show in the subsequent section that, for a wheel, the trivial inequalities and these three classes of inequalities completely describe the path packing polytope.

If $c^T x \geq \gamma$ is a valid inequality for the polytope $\mathrm{PP}\,(W, \mathcal{T})$, every path packing P such that $c^T \chi^P = \gamma$ is called a *root* (of the inequality $c^T x \geq \gamma$). If, in addition, the path packing P is edge-minimal, we say that P is an edge-minimal root.

Obviously, the whole edge set E and, for every $e \in E$, the set $E \setminus \{e\}$ are path packings in W. The incidence vectors of these edge sets are affinely independent. Hence, $\mathrm{PP}\,(W, \mathcal{T})$ is full dimensional, i.e., $\dim(\mathrm{PP}\,(W, \mathcal{T})) = |E|$.

Let us now introduce some inequalities that define facets for $\mathrm{PP}\,(W, \mathcal{T})$.

Obviously, the *trivial inequalities* $x_e \geq 0$ and $x_e \leq 1$, $e \in E$, are valid for the path packing polytope. It is also easy to show that they define facets for $\mathrm{PP}\,(W, \mathcal{T})$.

Let U be a node set that is an interval on the cycle C and that contains exactly one terminal, say l_{i_0} or r_{i_0}. The inequality

$$x(\delta(U)) \geq 1,$$

called *1-cut inequality*, is valid for $\mathrm{PP}\,(W,\mathcal{T})$, since every packing of paths P connects l_{i_0} to r_{i_0} and, hence, at least one edge from the cut $\delta(U)$ must be used.

Now, let U be an interval on the cycle C that contains exactly two terminals that do not form a terminal pair, i.e., there is an index i_0 such that $U \cap \bigcup_{i=1}^{k}\{l_i, r_i\} = \{r_{i_0}, l_{i_0+1}\}$. The inequality

$$x(\delta(U)) \geq 2,$$

called *2-cut inequality*, is valid for $\mathrm{PP}\,(W,\mathcal{T})$, since every packing of paths P connects r_{i_0} to l_{i_0} and l_{i_0+1} to r_{i_0+1}. Hence, at least two edges from the cut $\delta(U)$ must be used.

All 1-cut and 2-cut inequalities define facets of $\mathrm{PP}\,(W,\mathcal{T})$. The proofs of these facts are straight-forward, so we omit them. The number of different 1-cut and 2-cut inequalities is at most $O(n^2)$.

Let us now turn to the windmill inequalities.

Definition 3.1 *For $i = 1,\dots,k$, choose an edge set $F_i \subseteq C(l_i : r_i)$ with $1 \leq |F_i| \leq 2$ and some node $u_i^0 \in [r_i : l_{i+1}]$. We define a vector $a := a(F_1,\dots,F_k, u_1^0,\dots,u_k^0) \in \mathbb{R}^E$ by*

$$a_e = \begin{cases} 2, & \text{if } \{e\} = F_i \text{ for some } i \in \{1,\dots,k\},\\ 2, & \text{if } e = zv \text{ with } v \in [r_i : l_{i+1}] \setminus \{u_i^0\} \text{ for some } i \in \{1,\dots,k\},\\ 2, & \text{if } e = zv \text{ with } v \in [l_i : r_i] \setminus \{l_i, r_i\} \text{ for some } i \in \{1,\dots,k\}\\ & \text{and } C(l_i : v) \cap F_i = \emptyset \text{ or } C(v : r_i) \cap F_i = \emptyset,\\ 0, & \text{if } e = zu_i^0 \text{ for some } i \in \{1,\dots,k\} \text{ or}\\ & \text{if } e \in C \setminus \cup_{i=1}^{k} F_i,\\ 1, & \text{otherwise.} \end{cases}$$

The inequality

$$a(F_1,\dots,F_k,u_1^0,\dots,u_k^0)^T x \geq 2\lceil\frac{k}{2}\rceil$$

is called windmill inequality.

For an illustration of a windmill inequality, see Figure 4. The coefficients of a windmill inequality are determined by the following principles. For every interval whose endnodes form a terminal pair, we choose one or two special edges contained in this interval. If we choose one edge the corresponding component of a is set to 2, if we choose two edges the corresponding components of a are set to 1; the components of a corresponding to the other edges of the interval are set to 0. Moreover, for every edge of the outer cycle C that does not belong to such an

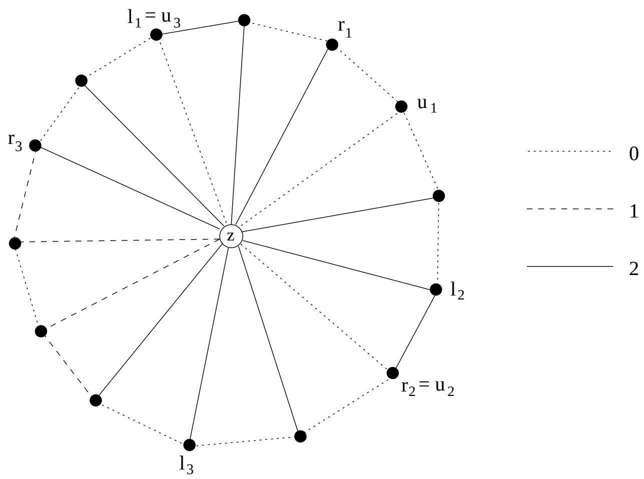

Figure 4:

interval the corresponding component of a is also set to 0. The coefficients corresponding to spokes can be determined as follows. From every interval $[r_i : l_{i+1}]$ (we say that $[r_i : l_{i+1}]$ forms a consecutive mixed interval) we choose a node u_i^0. The coefficient of a corresponding to the spoke zu_i^0 is set to 0. If $u_{i+1}^0 = u_i^0 + 1$ then there are no spokes between u_i^0 and u_{i+1}^0. Otherwise, the coefficients of the spokes $S(u_i^0 + 1 : u_{i+1}^0 - 1)$ of the open fan $F(u_i^0 : u_{i+1}^0)$ are computed in the following way. For every $v \in [u_i^0 + 1 : u_{i+1}^0 - 1]$, let Q_l and Q_r denote the path from v to l_{i+1} and from v to r_{i+1}, respectively, using edges only of $C(u_i^0 : u_{i+1}^0)$. Then $a_{vz} := \max\{\sum_{e \in Q_l} a_e, \sum_{e \in Q_r} a_e\}$.

Note that, if in Definition 3.1 all edge sets $F_i \, (i = 1, \ldots, k)$ have cardinality 1, the windmill inequality coefficients are zero or two, so it can be devided by two to obtain an inequality in standard coprime form. In this case, we speak of the *1-windmill inequality*, otherwise of the *2-windmill inequality*. Figure 5 illustrates an example of a 1-windmill inequality.

Lemma 3.2 *The windmill inequalities are valid for $PP(W, \mathcal{T})$.*

Proof. We start with the 1-windmill inequalities. For $i = 1, \ldots, k$, let $F_i := \{[t_i, t_i + 1]\} \subseteq C(l_i : r_i)$ and $u_i \in [r_i : l_{i+1}]$ be given. Then, by summing up the 2-cut inequalities $x(\delta([t_i + 1 : t_{i+1}])) \geq 2$ and the trivial inequalities $-x_{zu_i} \geq -1$, for $i = 1, \ldots, k$, dividing the resulting inequality by 2 and rounding the right hand side and the coefficients of the left hand side up, we obtain the 1-windmill inequality $\frac{1}{2} a(F_1, \ldots, F_k, u_1, \ldots, u_k)^T x \geq \lceil \frac{k}{2} \rceil$.

Now consider a 2-windmill inequality. For $i = 1, \ldots, k$, let $F_i = \{[t_i^1, t_i^1 + 1], [t_i^2, t_i^2 + 1]\} \subseteq C(l_i : r_i)$ and $u_i \in [r_i : l_{i+1}]$ be given, where, in case $|F_i| = 1$,

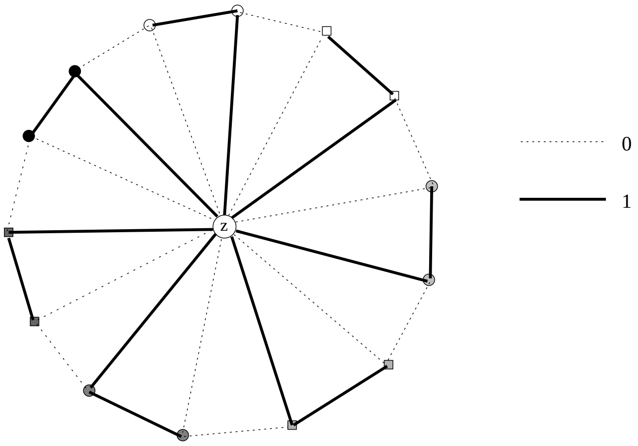

Figure 5:

the nodes t_i^1 and t_i^2 coincide. We sum up the following inequalities

$$\begin{array}{rll}
\frac{1}{2}a(\{[t_1^1, t_1^1+1]\}, \ldots, \{[t_k^1, t_k^1+1]\}, u_1, \ldots, u_k)^T x & \geq \lceil \frac{k}{2} \rceil, & \\
\frac{1}{2}a(\{[t_1^2, t_1^2+1]\}, \ldots, \{[t_k^2, t_k^2+1]\}, u_1, \ldots, u_k)^T x & \geq \lceil \frac{k}{2} \rceil, & \\
x(\delta([t_i^2+1 : t_{i+1}^1])) & \geq 2, & \text{for } i = 1, \ldots, k, \\
-x_{zu_i} & \geq -1, & \text{for } i = 1, \ldots, k.
\end{array}$$

Dividing the resulting inequality by 2 and rounding the right hand side and the coefficients of the left hand side up, results in the 2-windmill inequality $a(F_1, \ldots, F_k, u_1, \ldots, u_k)^T x \geq 2\lceil \frac{k}{2} \rceil$. □

The proof of Lemma 3.2 shows that windmill inequalities do not define facets of PP $(W, \mathcal{T})$, if k is even. However, in case k is odd, they do. The proof follows by standard arguments and is easy but lengthy. We thus refrain from stating the details.

Summing up our discussions, we have shown that the path packing polytope PP $(W, \mathcal{T})$ is contained in the polytope that is described by the trivial inequalities, the 1- and 2-cut inequalities and the windmill inequalities. In the subsequent section we will prove that both polytopes are equal.

4 A Complete Description of PP $(W, \mathcal{T})$

In this section we show that the inequalities introduced in the last section, i. e., the trivial inequalities, the 1- and 2-cut inequalities, and the windmill inequalities, completely describe the polytope PP $(W, \mathcal{T})$, if W is a wheel and $\mathcal{T}$ a list of consecutive terminal pairs. We prove this in two steps. First, we show that every facet-defining inequality that is not a trivial or a cut inequality has a number of properties:

Theorem 4.1 *Let $W = (V, E)$ be a wheel and $\mathcal{T} = \{\{l_1, r_1\}, \ldots, \{l_k, r_k\}\}$ a list of consecutive terminal pairs. Let $c^T x \geq \gamma$ be a facet-defining inequality of $PP(W, \mathcal{T})$ that is neither trivial nor a cut inequality. Then $c^T x \geq \gamma$ satisfies the following:*

(a) $c \geq 0$ *and* $\gamma > 0$.

(b) $c_e = 0$ *for all* $e \in C(r_i : l_{i+1})$, $i = 1, \ldots, k$.

(c) *For every* $i = 1, \ldots, k$, *there exists exactly one node* $u_i^0 \in [r_i : l_{i+1}]$ *with* $c_{zu_i^0} = 0$.

(d) $c_{zu} = \max\{c(C(l_i : u)), c(C(u : r_i))\}$, *for all* $u \in [l_i : r_i] \setminus \{l_i, r_i\}$, $i = 1, \ldots, k$.

(e) $c_{zu} = c(C(l_i : r_i))$, *for all* $u \in [u_{i-1}^0 : l_i] \setminus \{u_{i-1}^0\}$ *and all* $u \in [r_i : u_i^0] \setminus \{u_i^0\}$, $i = 1, \ldots, k$.

(f) $c(C(l_i : r_i)) = c(C(l_j : r_j))$, *for all* $i, j = 1, \ldots, k$.

(g) $\gamma = \lceil \frac{k}{2} \rceil \cdot c(C(l_1 : r_1))$.

The subsequent Lemmas 4.2 through 4.11 collectively prove Theorem 4.1. In the second step, see Theorem 4.12, we show that every inequality that satisfies the properties of Theorem 4.1 is a nonnegative linear combination of windmill inequalities. This shows that, indeed, the trivial inequalities, the 1- and 2-cut inequalities, and the windmill inequalities provide a complete description of PP $(W, \mathcal{T})$.

We suppose from now on that $c^T x \geq \gamma$ is a facet-defining inequality that is not a trivial or a cut inequality. Set $F_c := \{x \in \text{PP}\,(W, \mathcal{T}) \mid c^T x = \gamma\}$. Recall that, for each edge-minimal path packing P there is a unique path partition $P_1, \ldots, P_k$ of P satisfying the properties of Lemma 2.2. Then, the following lemmas hold.

Lemma 4.2 *Theorem 4.1 (a) is true.*

Proof. For each $e \in E$, there exists a root P with $e \notin P$, otherwise F_c would be contained in the face induced by the trivial inequality $x_e \leq 1$. Then, $P' := P \cup \{e\}$ is also a path packing with $c^T(\chi^{P'}) \geq \gamma$, and we obtain $0 \leq c^T(\chi^{P'}) - c^T(\chi^P) = c_e$. Moreover, since $c^T x \geq \gamma$ is facet-defining and not one of the trivial inequalities $x_e \geq 0$, $e \in E$, we conclude that $\gamma > 0$. $\square$

Lemma 4.3 *Theorem 4.1 (b) is true.*

Proof. Suppose Theorem 4.1 (b) does not hold. Then, there exist indices $i \in \{1, \ldots, k\}$ and $r \in [r_i : l_{i+1} - 1]$ such that $c_{[r,r+1]} > 0$. We pick one such i and select r as follows. If $c_{[r_i,r_i+1]} > 0$, we choose $r := r_i$, otherwise we choose r such that $c_{[s,s+1]} = 0$, for all $s \in [r_i : r-1]$. Set $e := [r, r+1]$. Since $c^T x \geq \gamma$ is a nontrivial facet-defining inequality, there exists an edge-minimal root P with $e \in P$. W. l. o. g. we assume that $e \in P_i$ (the other case $e \in P_{i+1}$ can be shown analogously). From Lemma 2.2 (ii) we know that there exists a node $t_0 \in [r+1 : l_{i+1}]$ with $zt_0 \in P_i$. Thus,

$$(*) \qquad c_{zt} \geq c_e + c_{zt_0} > 0, \quad \text{for all } t = r_i, \ldots, r.$$

Moreover, there exists a node $p \in [l_i : r_i - 1]$ with $c_{[p,p+1]} > 0$, since $c_e > 0$. Among all such nodes we choose the right-most node, i. e., we choose $p := r_i - 1$, if $c_{[r_i-1,r_i]} > 0$, otherwise we choose p such that $c_{[p',p'+1]} = 0$ for all $p' \in [p+1 : r_i - 1]$. Furthermore, the choice of p and $c_e > 0$ imply in case $p \neq r_i - 1$ that

$$c_{zt} \geq c_e + c_{zt_0} > 0, \quad \text{for all } t = p+1, \ldots, r_i - 1.$$

Summing up, we conclude that $c_f > 0$, for all $f \in \delta([p+1 : r])$. Since $c^T x \geq \gamma$ is a facet-defining inequality that is not a 1-cut inequality, there exists an edge-minimal root P^* with $\chi^{P^*} \notin \{x \in \mathrm{PP}\,(W, \mathcal{T}) \mid x(\delta([p+1 : r])) = 1\}$, i. e., $|P^* \cap \delta([p+1 : r])| \geq 2$. The facts that P^* is an edge-minimal root and that $c^T x \geq \gamma$ is valid imply that $e \in P^*_{i+1}$. Lemma 2.2 (ii) implies that there exists a node $t_1 \in [r_i : r]$ with $zt_1 \in P^*_{i+1}$. Thus,

$$c_{zt} \geq c_e + c_{zt_1} > 0, \quad \text{for all } t = r+1, \ldots, l_{i+1}.$$

Together with $(*)$, we obtain that $c_{zt_0} \geq c_e + c_{zt_1} \geq 2c_e + c_{zt_0}$. This relation and Theorem 4.1 (a) imply $c_e = 0$, a contradiction. $\square$

Lemma 4.4 *For all $i = 1, \ldots, k$, there exists a node $u \in [r_i : l_{i+1}]$ with $c_{zu} = 0$.*

Proof. Suppose, there exists an index $i \in \{1, \ldots, k\}$ such that $c_{zu} > 0$, for all $u \in [r_i : l_{i+1}]$. We prove that, in this case, $c^T x \geq \gamma$ is a multiple of a 2-cut inequality. First, we show that there is a positive edge on the path $C(l_i : r_i)$. Since $c^T x \geq \gamma$ is a nontrivial facet-defining inequality, there exists a root P with $[r_i - 1, r_i] \notin P$. Therefore, $c(P_i) > 0$. Obviously, $P' := P \setminus P_i \cup C(l_i : r_i)$ is also a packing of paths where $0 \leq c^T \chi^{P'} - c^T \chi^P = c(C(l_i : r_i)) - c(P_i)$. Thus, $c(C(l_i : r_i)) \geq c(P_i) > 0$. Analogously, we obtain that $c(C(l_{i+1} : r_{i+1})) > 0$. Among all nodes p_i in $[l_i : r_i - 1]$ such that $c_{[p_i,p_i+1]} > 0$ we choose the right-most node, i. e., if $c_{[r_i-1,r_i]} > 0$, we choose $p_i := r_i - 1$, otherwise we choose p_i such that $c_{[p',p'+1]} = 0$ for all $p' \in [p_i + 1 : r_i - 1]$. Similarly, among all nodes p_{i+1} in $[l_{i+1} : r_{i+1} - 1]$ such that $c_{[p_{i+1},p_{i+1}+1]} > 0$ we choose the left-most node, i. e., if $c_{[l_{i+1},l_{i+1}+1]} > 0$ we choose $p_{i+1} := l_{i+1}$, otherwise we choose p_{i+1} such

that $c_{[p',p'+1]} = 0$ for all $p' \in [l_{i+1} : p_{i+1} - 1]$. We now show that all edges in $\delta([p_i + 1 : p_{i+1}])$ are positive. If $p_i \neq r_i - 1$, consider a node $u \in [p_i + 1 : r_i - 1]$ and let $f \in S(r_i : l_{i+1}) \cap P$. Obviously, $\bar{P} := P \setminus \{f\} \cup (C(u : r_i) \cup \{zu\})$ is also a path packing. Due to Theorem 4.1 (b) and the choice of p_i we obtain that $0 \leq c^T\chi^{\bar{P}} - c^T\chi^P = c_{zu} - c_f$. Hence, $c_{zu} \geq c_f > 0$. Analogously, if $p_{i+1} \neq l_{i+1}$, we get that $c_{zu} > 0$, for all $u \in [l_{i+1} + 1 : p_{i+1}]$. Summing up, we conclude that $c_e > 0$, for all $e \in \delta([p_i + 1 : p_{i+1}])$.
Now, consider any root P^*. It is easy to check that $|P_i^* \cap \delta([p_i+1 : p_{i+1}])| = 1$ and that $|P_{i+1}^* \cap \delta([p_i+1 : p_{i+1}])| = 1$. From Lemma 2.2 we know that $|P_t^* \cap \delta([p_i+1 : p_{i+1}])| = 0$ for all $t \in \{1, \ldots, k\} \setminus \{i, i+1\}$. Therefore, $c^Tx \geq \gamma$ is a multiple of the 2-cut inequality $x(\delta([p_i + 1 : p_{i+1}])) \geq 2$, a contradiction. □

In the following we denote, for $i = 1, \ldots, k$, by $P^i_{min} \subseteq F(l_i : r_i)$ a path from l_i to r_i such that $c(P^i_{min}) = \min\{c(H) \mid H$ is a path from l_i to r_i with $H \subseteq F(l_i : r_i)\}$.

Lemma 4.5 *Consider an index $i \in \{1, \ldots, k\}$. If $c(P^i_{min}) > 0$, then*

- *$c_{zu} = 0$, for at most one $u \in [r_{i-1} : l_i]$.*
- *$c_{zu} = 0$, for at most one $u \in [r_i : l_{i+1}]$.*

Proof. Let $U_{i-1} := \{u \in [r_{i-1} : l_i] \mid c_{zu} = 0\}$ and $U_i := \{u \in [r_i : l_{i+1}] \mid c_{zu} = 0\}$. Since $c(P^i_{min}) > 0$ and because of Theorem 4.1 (b), it is easy to check that it is impossible that both $|U_{i-1}| \geq 2$ and $|U_i| \geq 2$ hold. Suppose w. l. o. g. that $|U_{i-1}| \geq 2$ and $|U_i| = 1$, say $u_{i-1}, v \in U_{i-1}$ with $v \in [u_{i-1} + 1 : l_i]$ and $U_i = \{u_i\}$. We use this assumption to construct from a root P of $c^Tx \geq \gamma$ a path packing $\bar{P}$ with $c^T\chi^{\bar{P}} < \gamma$, which contradicts the validity of $c^Tx \geq \gamma$. Since $c(P^i_{min}) > 0$, there exists a node $p \in [l_i : r_i - 1]$ with $c_{[p,p+1]} > 0$. We consider two cases:
(a) $p = r_i - 1$. Since $c^Tx \geq \gamma$ is a nontrivial facet-defining inequality, there exists a minimal root P with $zu_i \notin P$. Then, we know that $c(P_i) > 0$ and that $P_{i+1} \cap C(r_i : u_i) = \emptyset$. Moreover, $P_{i-1} \cap C(v : l_i) = \emptyset$ and $zv \notin P_{i-1}$, since $c_{zu_i-1} = 0$ and $v \in [u_{i-1} + 1 : l_i]$. This means that $\bar{P} := P \setminus P_i \cup (C(r_i : u_i) \cup \{zu_i, zv\} \cup C(v : l_i))$ is also a path packing with $c(\bar{P}) = c(P) - c(P_i) < \gamma$, a contradiction.
(b) $p \neq r_i - 1$. Let $H^* \subseteq F[p + 1 : r_i)$ be a path from r_i to z such that $c(H^*) = \min\{c(H) \mid H \subseteq F[p + 1 : r_i), H$ is a path from r_i to $z\}$. In case $c(H^*) > 0$ we obtain a contradiction by the same construction as in (a). Suppose, $c(H^*) = 0$. Since $c^Tx \geq \gamma$ is a nontrivial facet-defining inequality, there exists an edge-minimal root P^* with $[p, p + 1] \in P^*$. Thus, $c(P_i^*) > 0$ and we can assume w. l. o. g. that $P_{i-1}^* \cap C(v : l_i) = \emptyset$ and $zv \notin P_{i-1}^*$, since $c_{zu_i-1} = 0$ and $v \in [u_{i-1} + 1 : l_i]$. Then, $\bar{P} := P^* \setminus P_i^* \cup (H^* \cup \{zv\} \cup C(v : l_i))$ is also a path packing with $c(\bar{P}) = c(P^*) - c(P_i^*) < \gamma$, a contradiction.
Summing up, both cases lead to a contradiction, and we conclude that $|U_{i-1}| = |U_i| = 1$. □

Lemma 4.6 *Consider an index $i \in \{1,\dots,k\}$. If $c(S(r_i : l_{i+1})) > 0$, then $c(P^i_{min}) > 0$ and $c(P^{i+1}_{min}) > 0$.*

Proof. Let $v \in [r_i : l_{i+1}]$ with $c_{zv} > 0$. From Lemma 4.4 we know that there exists a node $u \in [r_i : l_{i+1}]$ with $c_{zu} = 0$. W. l. o. g. we assume that $v \in [u : l_{i+1}]$ (the other case $u \in [v : l_{i+1}]$ can analogously be shown). Since $c^T x \geq \gamma$ is a nontrivial facet-defining inequality, there exists an edge-minimal root P with $zv \in P$. If $zv \in P_i$, we get that $P' := P \setminus \{zv\} \cup \{zu\}$ is also a path packing with $c(P') < c(P) = \gamma$, a contradiction. Thus, we know that $zv \in P_{i+1}$. Since $P' := P \setminus P_{i+1} \cup P^{i+1}_{min}$ is also a path packing with $0 \leq c(P') - c(P) = c(P^{i+1}_{min}) - c(P_{i+1})$, we get that $c(P^{i+1}_{min}) \geq c(P_{i+1}) > 0$, since $zv \in P_{i+1}$. Now, suppose $c(P^i_{min}) = 0$. In this case we can assume w. l. o. g. that $P_i = P^i_{min}$. Then, $P' := P \setminus \{zv\} \cup (C(u : v) \cup \{uz\})$ is also a path packing with $c(P') < c(P) = \gamma$, a contradiction. □

Theorem 4.1 (c) can now be derived from Lemma 4.5 and Lemma 4.6: Since $\gamma > 0$, there exists an index i_0 with $c(P^{i_0}_{min}) > 0$. Applying Lemma 4.5 we conclude that $c(S(r_{i_0} : l_{i_0+1})) > 0$, since $|[r_{i_0} : l_{i_0+1}]| \geq 2$. From Lemma 4.6 we obtain that $c(P^{i_0+1}_{min}) > 0$ as well. Continuing this way, we get that $c(P^i_{min}) > 0$, for all $i = 1,\dots,k$. This together with Lemma 4.5 and Lemma 4.6 implies Theorem 4.1 (c).

In the following we denote by $u^0_i \in [r_i : l_{i+1}]$ the unique node with $c_{zu^0_i} = 0$, for $i = 1,\dots,k$. In order to prove Theorem 4.1 (d) we need the following lemma.

Lemma 4.7 *Consider an index $i \in \{1,\dots,k\}$. Let P be an edge-minimal root such that P_i contains at most one of the edges zu^0_{i-1} and zu^0_i. Then, $c(P_i) = c(C(l_i : r_i))$.*

Proof. First of all, note that, for all edge-minimal roots P, $c(P_i) \leq c(C(l_i : r_i))$, since $P \setminus P_i \cup C(l_i : r_i)$ is also a path packing. Now suppose, there exists an edge-minimal path packing P with $|\{zu^0_{i-1}, zu^0_i\} \cap P_i| \leq 1$ such that $c(P_i) < c(C(l_i : r_i))$. Obviously, $z \in V(P_i)$. Let $u, v \in [u^0_{i-1} : u^0_i]$ with $zu, zv \in P_i$. W. l. o. g. we can assume that $v \in [u^0_{i-1} : u]$ and $u \neq u^0_i$. Since $c^T x \geq \gamma$ is a nontrivial facet-defining inequality, there exists an edge-minimal root P' with $zu^0_{i-1} \notin P'$. If $P'_i = C(l_i : r_i)$, we have that $P^* := P' \setminus P'_i \cup P_i$ is also a path packing (note that $u \neq u^0_i$) with $c(P^*) = c(P') - c(P'_i) + c(P_i) < \gamma$, a contradiction. We conclude that $z \in V(P'_i)$. Now, consider the unique path H_{lz} in P'_i from l_i to z. Since $zu^0_{i-1} \notin P'$, we get that $c(H_{lz}) = 0$. This fact, however, means that there cannot exist a root $\bar{P}$ that contains the edge zw, for any $w \in [r_{i-1} : l_i] \setminus \{u^0_{i-1}\}$. Thus, $c^T x \geq \gamma$ is not a facet-defining inequality, a contradiction. □

Lemma 4.8 *Theorem 4.1 (d) is true.*

Proof. Let $i \in \{1, \dots, k\}$ be an index with $l_i + 1 \neq r_i$ and $u \in [l_i + 1 : r_i - 1]$ be given. Since $c^T x \geq \gamma$ is a nontrivial facet-defining inequality, there exists an edge-minimal root P with $zu_i^0 \notin P$. Due to Lemma 4.7 we can assume that $P_i = C(l_i : r_i)$. Then, $P^* := P \setminus P_i \cup (C(l_i : u) \cup \{zu, zu_i^0\} \cup C(r_i : u_i^0))$ is also a path packing with $0 \leq c(P^*) - c(P) = c_{zu} - c(C(u : r_i))$. Thus,

$$c_{zu} \geq c(C(u : r_i)). \tag{1}$$

Analogously, there exists an edge-minimal root $\bar{P}$ with $zu_{i-1}^0 \notin \bar{P}$, and we conclude

$$c_{zu} \geq c(C(l_i : u)). \tag{2}$$

Since $c(C(l_i : r_i)) \geq c(P_{min}^i) > 0$, it follows from (1) and (2) that $c_{zu} > 0$. Hence, there exists an edge-minimal root $\tilde{P}$ with $zu \in \tilde{P}_i$. Since $\tilde{P}$ is edge-minimal, either $C(l_i : u) \subset \tilde{P}_i$ or $C(u : r_i) \subset \tilde{P}_i$. In the first case, we conclude that $c(C(u : r_i)) \geq c_{zu}$, since $\tilde{P} \setminus \{zu\} \cup C(u : r_i)$ is also a path packing. This together with (1) implies $c_{zu} = c(C(u : r_i))$, and, because of (2), $c(C(u : r_i)) \geq c(C(l_i : u))$. In other words, $c_{zu} = \max\{c(C(u : r_i)), c(C(l_i : u))\}$. In the latter case (i. e., $C(u : r_i) \subset \tilde{P}_i$), we get that $c(C(l_i : u)) \geq c_{zu}$, since $\tilde{P} \setminus \{zu\} \cup C(l_i : u)$ is also a path packing. By the same arguments as in the first case we obtain $c_{zu} = \max\{c(C(u : r_i)), c(C(l_i : u))\}$ in this case as well. □

Lemma 4.9 *Theorem 4.1 (e) is true.*

Proof. Let $i \in \{1, \dots, k\}$ be an index with $r_i \neq u_i^0$ and $u \in [r_i : u_i^0 - 1]$. Since $c^T x \geq \gamma$ is a nontrivial facet-defining inequality and $c_{zu} > 0$ by Theorem 4.1 (c), there exists an edge-minimal root P with $zu \in P$. Moreover, $zu \in P_i$, because $u_i^0 \in [u + 1 : l_{i+1}]$. Then, $P^* := P \setminus \{zu\} \cup C(l_i : r_i)$ is also a path packing with $0 \leq c(P^*) - c(P) = c(C(l_i : r_i) \setminus P_i) - c_{zu}$. Thus, we have that $c(C(l_i : r_i)) \geq c(C(l_i : r_i) \setminus P_i) \geq c_{zu}$. Furthermore, there exists an edge-minimal root P' with $zu_{i-1}^0 \notin P'$. Due to Lemma 4.7 we can assume w. l. o. g. that $P'_i = C(l_i : r_i)$. Since $u_i^0 \in [u + 1 : l_{i+1}]$, we know that $zu \notin P'_{i+1}$, and thus $zu \notin P'$. This implies that $P^* := P' \setminus P'_i \cup (C(u_{i-1}^0 : l_i) \cup \{zu_{i-1}^0, zu\} \cup C(r_i : u))$ is also a path packing with $0 \leq c(P^*) - c(P') = c_{zu} - c(C(l_i : r_i))$. Thus, we also have that $c_{zu} \geq c(C(l_i : r_i))$, and we conclude that equality must hold. In an analogous way it can be shown that $c_{zu} = c(C(l_i : r_i))$ for all $u \in [u_{i-1}^0 + 1 : l_i]$, if $u_{i-1}^0 \neq l_i$. □

Lemma 4.10 *Theorem 4.1 (f) is true.*

Proof. Consider an index $i \in \{1, \dots, k\}$. We know that there exists an edge-minimal root P with $zu_{i-1}^0 \notin P$. Lemma 4.7 implies that we can assume w. l. o. g. that $P_i = C(l_i : r_i)$. This means that $zu_i^0 \in P_{i+1}$, since otherwise

$P' := P \setminus P_i \cup Q$, where $Q := C(u^0_{i-1} : l_i) \cup \{zu^0_{i-1}, zu^0_i\} \cup C(r_i : u^0_i)$, is a path packing with $c^T \chi^{P'} < \gamma$. Moreover, we conclude from Lemma 4.7 that $zu^0_{i+1} \in P_{i+1}$ and, thus, $c(P_{i+1}) = 0$. Hence, $P^* := P \setminus (P_i \cup P_{i+1}) \cup (Q \cup C(l_{i+1} : r_{i+1}))$ is also a packing of paths with $0 \leq c(P^*) - c(P) = c(C(l_{i+1} : r_{i+1})) - c(C(l_i : r_i))$. Thus, $c(C(l_{i+1} : r_{i+1})) \geq c(C(l_i : r_i))$. Iterating this argument proves Theorem 4.1 (f). $\square$

Lemma 4.11 *Theorem 4.1 (g) is true.*

Proof. First, we construct a packing of paths P whose value $c(P)$ is equal to $\lceil \frac{k}{2} \rceil \cdot c(C(l_1 : r_1))$. For $i = 1, \ldots, k$ we define

$$P_i := \begin{cases} C(l_i : r_i), & \text{if } i \text{ is odd,} \\ C(u^0_{i-1} : l_i) \cup \{zu^0_{i-1}, zu^0_i\} \cup C(r_i : u^0_i), & \text{if } i \text{ is even.} \end{cases}$$

It is easy to check that P_i is a path from l_i to r_i $(i = 1, \ldots, k)$ and that $P_1, \ldots, P_k$ are mutually disjoint. Thus, $P := \cup_{i=1}^k P_i$ is a packing of paths. By applying Lemma 4.10 we obtain that

$$\begin{aligned} c(P) &= \sum_{i \text{ odd}} c(P_i) + \sum_{i \text{ even}} c(P_i) \\ &= \sum_{i \text{ odd}} c(C(l_i : r_i)) \\ &= \lceil \tfrac{k}{2} \rceil \cdot c(C(l_1 : r_1)). \end{aligned}$$

This implies that $\gamma \leq \lceil \frac{k}{2} \rceil \cdot c(C(l_1 : r_1))$.
Now, consider any root P. Let $\pi_i := |P_i \cap \{zu^0_{i-1}, zu^0_i\}|$ for $i = 1, \ldots, k$. From Lemma 4.7 and Lemma 4.10 we know that $c(P_i) = c(C(l_i : r_i)) = c(C(l_1 : r_1))$, if $\pi_i \leq 1$. On the other hand, the number of indices $i \in \{1, \ldots, k\}$ with $\pi_i = 2$ is at most $\lfloor \frac{k}{2} \rfloor$. Thus, $\gamma = c(P) \geq \sum_{\{i | \pi_i \leq 1\}} c(P_i) = \sum_{\{i | \pi_i \leq 1\}} c(C(l_1 : r_1)) \geq \lceil \frac{k}{2} \rceil \cdot c(C(l_1 : r_1))$. $\square$

In the following theorem we show that each inequality $c^T x \geq \gamma$ that satisfies Theorem 4.1 is a nonnegative linear combination of windmill inequalities.

Theorem 4.12 *Let $c^T x \geq \gamma$, $c \in \mathbb{Z}^E$, be an inequality satisfying Theorem 4.1. Then, there exists a set of windmill inequalities $a_i^T x \geq \alpha_i$ $(i = 1, \ldots, l)$ such that $\lambda \sum_{i=1}^l a_i = c$ and $\lambda \sum_{i=1}^l \alpha_i = \gamma$, where $\lambda = \frac{1}{2}$, if $c(C(l_1 : r_1))$ is odd, and $\lambda = 1$, otherwise.*

Proof. Let $c^T x \geq \gamma$ be an inequality satisfying Theorem 4.1. By appropriate scaling of c we can assume that $c(C(l_i : r_i))$ is even. It is thus sufficient to prove

Theorem 4.12 for all integral inequalities $c^T x \geq \gamma$ with $c(C(l_1 : r_1))$ even. We show this by induction on $\eta := c(C(l_1 : r_1))$.

η is positive because of Theorem 4.1 (a) and (g). If $\eta = 2$, $c^T x \geq \gamma$ is obviously a windmill inequality, see Definition 3.1 and the explanation thereafter.

Now let $\eta \geq 4$. We suppose that Theorem 4.12 is true for all inequalities $b^T x \geq \beta$ that satisfy Theorem 4.1, and for which $b(C(l_1 : r_1)) < \eta$ and even. In the following we construct a windmill inequality. For $i = 1, \ldots, k$, let $U_i := \{uv \in C(l_i : r_i) \mid c_{uv} > 0\}$. Suppose, $U_i = \{e_1, \ldots e_s\}$, $s \geq 1$, where $e_1, \ldots, e_s$ are numbered in clockwise order by walking from l_i to r_i. If $s = 1$, set $F_i := U_i$, otherwise set $F_i := \{e_1, e_s\}$. Then,

$$a(F_1, \ldots, F_k, u_1^0, \ldots, u_k^0)^T x \geq 2 \cdot \lceil \frac{k}{2} \rceil$$

is a windmill inequality. Let $a_0 := a(F_1, \ldots, F_k, u_1^0, \ldots, u_k^0)$ and $\alpha_0 := 2 \cdot \lceil \frac{k}{2} \rceil$, and set $b := c - a_0$ and $\beta := \gamma - \alpha_0$. We show that $b^T x \geq \beta$ satisfies Theorem 4.1 (a) through (g). Theorem 4.1 (a) to (c) hold by construction (note that $\beta > 0$, since $\eta \geq 4$). Moreover, $b(C(l_i : r_i)) = c(C(l_i : r_i)) - 2$, for all $i = 1, \ldots, k$ and, for all $u \in [u_{i-1}^0 + 1 : u_i^0 - 1]$, we have that

$$b_{zu} = \begin{cases} c_{zu} - 2, & \text{if } u_{i-1}^0 \neq l_i \text{ and } u \in [u_{i-1}^0 + 1 : l_i], \\ c_{zu} - 2, & \text{if } u_i^0 \neq r_i \text{ and } u \in [r_i : u_{i-1}^0 - 1], \\ c_{zu} - 2, & \text{if } l_i \neq r_i - 1,\ u \in [l_i + 1 : r_i - 1] \text{ and} \\ & c(C(l_i : u)) = 0 \text{ or } c(C(u : r_i)) = 0, \\ c_{zu} - 1, & \text{otherwise.} \end{cases}$$

This obviously shows Theorem 4.1 (d) to (f). Finally, $\beta = \gamma - 2 \cdot \lceil \frac{k}{2} \rceil = \lceil \frac{k}{2} \rceil \cdot c(C(l_1 : r_1)) - 2 \cdot \lceil \frac{k}{2} \rceil = (c(C(l_1 : r_1)) - 2) \cdot \lceil \frac{k}{2} \rceil = b(C(l_1 : r_1)) \cdot \lceil \frac{k}{2} \rceil$, which yields Theorem 4.1 (g). Since $b(C(l_1 : r_1)) < \eta$ and even, there exists, by induction hypothesis, a set of windmills $a_i^T x \geq \alpha_i$, $i = 1, \ldots, l$ such that $\sum_{i=1}^l a_i = b$ and $\sum_{i=1}^l \alpha_i = \beta$. Summing up, we obtain that $c = b + a_0 = \sum_{i=0}^l a_i$ and $\sum_{i=0}^l \alpha_i = \alpha_0 + \sum_{i=1}^l \alpha_i = \alpha_0 + \beta = \gamma$. □

Summarizing the results presented in Section 3 and 4 we have shown the following.

Theorem 4.13 *Let $W = (V, E)$ be a wheel with nonnegative edge lengths $w_e \in \mathbb{R}$, $e \in E$, and let $\mathcal{T} = \{\{l_1, r_1\}, \ldots, \{l_k, r_k\}\}$ be a list of consecutive terminal pairs. Then, for k even, a complete and nonredundant linear description of the path packing polytope $PP(W, \mathcal{T})$ is given by the following system of inequalities:*

Trivial inequalities: $0 \leq x_e \leq 1$ *for all* $e \in E$.

1-cut inequalities: $x(\delta(U)) \geq 1$ *for all intervals U of the outer cycle C of W containing exactly one of the terminals $\{l_1, r_1, \ldots, l_k, r_k\}$.*

2-cut inequalities: $x(\delta(U)) \geq 2$ *for all intervals* U *of the outer cycle* C *of* W *containing exactly two terminals of* $\{l_1, r_1, \ldots, l_k, r_k\}$ *that do not form a terminal pair.*

If k *is odd, the following inequalities are needed in addition.*

Windmill inequalities: $a(F_1, \ldots, F_k, u_1^0, \ldots, u_k^0)^T x \geq 2\lceil \frac{k}{2} \rceil$, *for all edge sets* $F_i \subseteq C(l_i : r_i)$ *with* $1 \leq |F_i| \leq 2$ *and all nodes* $u_i^0 \in [r_i : l_{i+1}]$ $(i = 1, \ldots, k)$ *and with* $a(F_1, \ldots, F_k, u_1^0, \ldots, u_k^0) \in \mathbb{R}^E$ *as in Definition 3.1.*

We remark that Theorem 4.13 can be generalized slightly. Namely, we also have a complete description of the path packing polytope (given a set of consecutive terminal pairs on the outer cycle) if, in the underlying wheel, every edge is replaced by a path (of arbitrary length). The polynomial time algorithm of Section 2 can trivially be adapted.

Final Remarks

To our knowledge, the algorithm presented in this paper for the minimum length path packing problem on wheels with consecutive terminal sets is one of very few (strongly) polynomial time algorithms for the optimization version of a path packing problem. It would be interesting to find extensions to more general or different cases. For instance, can one replace wheels by planar graphs or some class of planar graphs more general than wheels? Can one allow crossing terminal pairs on the outer face? Certainly, not in general, since even the existence of path packings cannot be shown in polynomial time unless additional evenness or other additional conditions such as in the Okamura-Seymour theorem are added. What about shortest tree or Steiner tree packings?

Our complete (and nonredundant) description of the path packing polytope for wheels with consecutive terminal pairs is a first step towards establishing a closer link between path packing theory and polyhedral combinatorics. We do not know any other result of this type and ask, similarly, for possible generalizations of the class of wheels and the properties of terminal pairs that allow explicit complete descriptions of the associated packing polytope. We were quite surprised when we discovered that in the case of an even number of terminal pairs the trivial and the 1-cut and 2-cut (and thus a polynomial number of inequalities) suffice but that for an odd number of terminal pairs a new class of inequalities, which we call windmill inequalities and that grows exponentially with the number of terminal pairs, is necessary in addition. Maybe more surprises and large classes of computationally useful inequalities are waiting for their discovery.

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