

Dissipativity enforcement via Perturbation of para-Hermitian Pencils

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Abstract

Dissipativity is an important property of individual systems that guarantees a stable interconnected system. However, due to errors in the modeling process weakly non-dissipative models may be constructed. In this paper we introduce a method to perturb a non-dissipative LTI system in order to enforce dissipativity using spectral perturbation results for para-Hermitian pencils. Compared to earlier algorithms the new method is applicable to a wider class of problems, it utilizes a simpler framework, and employs a larger class of allowable perturbations resulting in smaller perturbations. Moreover, system stability can be enforced as well. Numerical examples are provided to show the effectiveness of the new approach.

Index Terms

passivity, dissipativity, para-Hermitian matrices, eigenvalue perturbation, linear system, behavioral approach

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I. INTRODUCTION

THE RESEARCH that motivated this work started from an attempt to generalize the results of [1] to so called *descriptor systems*, i.e., linear time-invariant systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad (1a)$$

$$y(t) = Cx(t) + Du(t), \quad (1b)$$

where $E, A \in \mathbb{R}^{\rho, n}$, $B \in \mathbb{R}^{\rho, m}$, $C \in \mathbb{R}^{\ell, n}$, and $D \in \mathbb{R}^{\ell, m}$. Conventionally, E and A are square, i.e., $\rho = n$, but non-square systems make sense in some situations, see Example 10. With \mathcal{C}_∞^q denoting the infinitely often differentiable functions mapping \mathbb{R} to \mathbb{R}^q we call $u \in \mathcal{C}_\infty^m$ the *input*, $x \in \mathcal{C}_\infty^n$ the *state*, and $y \in \mathcal{C}_\infty^\ell$ the *output*. A triple $(u, x, y) \in \mathcal{C}_\infty^m \times \mathcal{C}_\infty^n \times \mathcal{C}_\infty^\ell = \mathcal{C}_\infty^{m+n+\ell}$ which fulfills (1) is called a *trajectory* of (1). In [1] only the special case of (1) in which $\rho = n$ and $E = I$ (the identity matrix) was considered. However, the modeling of electrical circuits and systems often leads to systems of the form (1), where E is not the identity.

It is common practice [2] to measure the power supplied to a system of the form (1) at a time point t along a certain trajectory (u, x, y) through a so called *supply function* $s : \mathbb{R}^\ell \times \mathbb{R}^m \rightarrow \mathbb{R}$ of the form

$$s(u(t), y(t)) := \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}, \quad (2)$$

where $Q = Q^T \in \mathbb{R}^{\ell, \ell}$, $S \in \mathbb{R}^{\ell, m}$, and $R = R^T \in \mathbb{R}^{m, m}$. Consequently, the energy which is supplied to system (1) along a certain trajectory in the time interval $[t_0, t_1]$ is measured by

$$\int_{t_0}^{t_1} s(u(t), y(t)) dt.$$

In the well known special case of (1) describing an RLC circuit with current and voltage sources (e.g., [3]) we have $\rho = n$, $\ell = m$ and the supplied power is measured as the product of current and voltage, i.e., $s(u, y) = u^T y$ which corresponds to

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \quad (3)$$

If, however, (1) stems from a realization procedure of scattering parameters (e.g., [3]), we usually have $\rho = n$, $E = I$, and the supplied power is measured via $s(u, y) = \|u\|_2^2 - \|y\|_2^2$, i.e.,

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}. \quad (4)$$

The supplies given by (3) and (4) are essentially those, which are considered in earlier literature.

Colloquially speaking, a system is dissipative, if it cannot output more energy than is fed into it. A rigorous definition follows.

Definition 1. We call (1) dissipative (with respect to (2)) if the inequality

$$0 \leq \int_{-\infty}^{\infty} s(u(t), y(t)) dt,$$

holds for all trajectories (u, x, y) of (1) which have compact support, i.e., for all trajectories that vanish outside of a bounded interval.

For stable systems (1) with $E = I$, the well-known concepts of *passivity* and *contractivity* [2] are equivalent to dissipativity with respect to the special supply functions (3) and (4), respectively.

In this paper we consider the following question. Assume that (1) models a physical system which is known to be dissipative, i.e., it does not generate energy. However, due to errors or inaccuracies during modeling, linearization, discretization, realization, model order reduction, or similar we end up with a slightly non-dissipative system. Then a post-correction method is desired to perturb (part of) the matrices A, B, C, D, E, Q, S , and R such that the perturbed system is dissipative.

Earlier approaches to this question are manifold. Methods for systems given in rational function form, which are derived from tabulated data, are considered in [4], [5], [6]. Methods for state space systems based on semi-definite programming are presented in [7], [8], [9], [10].

The converse problem of computing the smallest perturbation that makes a passive system non-passive is studied in [11]. Finally we mention [12] where a method similar to ours is used to compute the pseudo spectral abscissa of a matrix.

Our approach is related to a series of papers beginning with [1], where the basic methodology is introduced. The scope there is restricted to the two cases i) $E = I$, $D + D^T$ invertible and supply functions of the form (3), and ii) $E = I$, $\|D\|_2 < 1$ and supply functions of the form (4). The main computational cost in [1] is the computation of eigenvalues and -vectors of a Hamiltonian matrix. In [13] this approach was enhanced by the use of structure preserving algorithms for the Hamiltonian eigenvalue problem, resulting in a faster and more robust method. The method in [14] can deal with general matrices E and D . Central to the method are now the eigenvalues of a Hamiltonian/skew-Hamiltonian pencil. Explicit computation of spectral projectors is used to extract relevant subsystems. The latter point is alleviated in [15] where no projectors are needed and the required eigenvalues and eigenvectors are computed in a structure-preserving manner. All the previous methods restrict the perturbations

to the matrix C . We also mention [16], [17] where smallest perturbations to Hamiltonian matrices and the computation of the distance to bounded-realness are considered.

Our new method deals with a wider class of problems (general supply function, non-square E, A) and allows a larger class of perturbations (any chosen entries of A, B, C, D, E, Q, R, S can be perturbed). Moreover, not only dissipativity, but also stability can be enforced.

The paper is structured as follows. The remainder of this section introduces further notation. Section II describes how to check for dissipativity of a given system using a characterization of dissipative systems in terms of the spectrum of a family of Hermitian matrices. During the next four sections our method is introduced: Section III reviews a spectral perturbation result that is used later on, in Sections IV and V first order conditions on the wanted perturbation (that enforces dissipativity and stability) are derived and in Section VI all the pieces are combined to the final algorithm. Numerical experiments are described in Section VII. Section VIII adapts the method to the behavior approach while in Section IX we offer our conclusions.

A. Notation

Let $\mathbb{R}[\lambda]$ denote the set of *polynomials* in the indeterminate λ and let $\mathbb{R}[\lambda]_1$ denote the set of first-order polynomials, i.e., all polynomials of the form $\lambda a + b$, where $a, b \in \mathbb{R}$. Let $\mathbb{R}[\lambda]^{p,q}$ denote the set of *polynomial matrices*, i.e., the p -by- q matrices in which every entry is a polynomial and analogously, let $\mathbb{R}[\lambda]_1^{p,q}$ denote the set of first-order polynomial matrices, which are also called *pencils*. Let $\mathbb{R}(\lambda)$ denote the set of *rational functions*. Since the rational functions form a field one can speak of the *rank* of a matrix $R \in \mathbb{R}(\lambda)^{p,q}$, whose entries are rational functions. We denote this rank (over the field of rational functions) by $\text{rank}_{\mathbb{R}(\lambda)}(R)$. Since the polynomials are a subset of the rational functions, we can also attribute a rank to every polynomial matrix $P \in \mathbb{R}[\lambda]^{p,q}$, namely $\text{rank}_{\mathbb{R}(\lambda)}(P)$. We say that P is regular (over $\mathbb{R}(\lambda)$) if $p = q$ and $\text{rank}_{\mathbb{R}(\lambda)}(P) = p$, or equivalently, if $p = q$ and $\det(P(\lambda))$ is not identically zero. Using the Smith canonical form [18, p. 141, Theorem 3] one can show that

$$\max_{\lambda_0 \in \mathbb{C}} \text{rank}(P(\lambda_0)) = \text{rank}_{\mathbb{R}(\lambda)}(P). \quad (5)$$

Furthermore, the maximum in (5) is attained for almost all $\lambda_0 \in \mathbb{C}$ and there is only a finite set of points where the rank drops, i.e., where $\text{rank}(P(\lambda_0)) < \text{rank}_{\mathbb{R}(\lambda)}(P)$. We call those points the *zeros* of a polynomial matrix and denote the set of all zeros by

$$\zeta(P) := \{\lambda_0 \in \mathbb{C} \mid \text{rank}(P(\lambda_0)) < \text{rank}_{\mathbb{R}(\lambda)}(P)\}.$$

Note that for regular matrix polynomials the zeros of P coincide with its eigenvalues.

For a matrix polynomial $P \in \mathbb{R}[\lambda]^{p,q}$ we define its *para-conjugate transposed* $P^\sim \in \mathbb{R}[\lambda]^{q,p}$ through

$$P^\sim(\lambda) := P^*(-\bar{\lambda}) = P^T(-\lambda)$$

and we call a matrix $\mathcal{N} \in \mathbb{R}[\lambda]^{k,k}$ *para-Hermitian* if $\mathcal{N} = \mathcal{N}^\sim$. A para-Hermitian matrix is Hermitian on the imaginary

axis, i.e., for $\omega \in \mathbb{R}$ we have

$$(\mathcal{N}(i\omega))^* = \mathcal{N}(-i\omega) = \mathcal{N}(i\omega),$$

where i denotes the imaginary unit. Thus, for every $\omega \in \mathbb{R}$ the matrix $\mathcal{N}(i\omega)$ has k real eigenvalues, let us say π non-negative and ν negative. Then we define the *signsum function* through the expression

$$\eta(\mathcal{N}(i\omega)) = \pi - \nu, \quad (6)$$

i.e., the signsum function of a Hermitian matrix is the number of non-negative eigenvalues minus the number of negative ones.

II. CHECKING DISSIPATIVITY

In order to enforce dissipativity we introduce a suitable characterization of a dissipative system. Our approach is based on the following result.

Theorem 2. Consider system (1) with supply (2). Assume

$$\text{rank} \begin{bmatrix} i\omega E - A & -B \end{bmatrix} = \rho, \quad (7)$$

for all $\omega \in \mathbb{R}$. Set $k := \rho + 2\ell + n + m$ and define the *para-Hermitian pencil* $\mathcal{N} = \mathcal{N}^\sim \in \mathbb{R}[\lambda]_1^{k,k}$ by

$$\mathcal{N}(\lambda) := \lambda \mathcal{N} - M := \begin{bmatrix} 0 & 0 & 0 & \lambda E - A & -B \\ 0 & 0 & I & -C & -D \\ 0 & I & Q & 0 & S \\ -\lambda E^T - A^T & -C^T & 0 & 0 & 0 \\ -B^T & -D^T & S^T & 0 & R \end{bmatrix} \quad (8)$$

Then (1) is dissipative if and only if for all $\omega \in \mathbb{R}$ we have

$$\eta(\mathcal{N}(i\omega)) = n + m - \rho. \quad (9)$$

Proof: See [19]. ■

Because of (7) we see that the right hand side of (9) specifies the dimension of the co-kernel of $\begin{bmatrix} \lambda E - A & -B \end{bmatrix}$. Thus the right hand side of (9) can be considered as the number of free variables in (1a). This is the reason why in behavioral systems theory [20], this quantity is also called the "number of inputs", although, if $\rho \neq n$, this notion conflicts with the definition of inputs from (1), since in this case some of the variables in the state x can be interpreted as inputs.

The zeros of $\begin{bmatrix} \lambda E - A & -B \end{bmatrix}$ are also called the *uncontrollable modes* of system (1), see [20]. Thus, the assumption (7) is a kind of controllability condition, since it demands that there are no uncontrollable modes on the imaginary axis, although this will not play a prominent role in the following. Also note that, by using (5), we obtain that (7) implies that $\text{rank}_{\mathbb{R}(\lambda)} \begin{bmatrix} \lambda E - A & -B \end{bmatrix} = \rho$, i.e., that $\begin{bmatrix} \lambda E - A & -B \end{bmatrix}$ has full row rank over $\mathbb{R}(\lambda)$.

To understand how Theorem 2 can help to solve the problem of dissipativity enforcement, we introduce what we call the *spectral plot* of a para-Hermitian matrix polynomial $\mathcal{N} \in \mathbb{R}[\lambda]^{k,k}$. This is done in the following way. First, for the given \mathcal{N} we uniquely define the function $f_{\mathcal{N}} : \mathbb{R} \rightarrow \mathbb{R}^k$ such that for every $\omega \in \mathbb{R}$ the vector $f_{\mathcal{N}}(\omega)$ contains the k real eigenvalues of the Hermitian matrix $\mathcal{N}(i\omega)$ in non-increasing

order. Now consider the situation in Theorem 2. In this case, for every $\omega \in \mathbb{R}$ the matrix $\mathcal{N}(i\omega)$ has the form

$$\mathcal{N}(i\omega) = \begin{bmatrix} 0 & W \\ W^* & H \end{bmatrix},$$

with

$$W = \begin{bmatrix} 0 & i\omega E - A & -B \\ I & -C & -D \end{bmatrix}$$

and $H = H^T$ given by

$$H = \begin{bmatrix} Q & 0 & S \\ 0 & 0 & 0 \\ S^T & 0 & R \end{bmatrix}.$$

Assumption (7) implies that $\text{rank}(W) = \rho + \ell$. Thus, by congruence transformations we obtain

$$\begin{aligned} \mathcal{N}(i\omega) &\sim \begin{bmatrix} 0 & I_{\rho+\ell} & 0 \\ I_{\rho+\ell} & H_{11} & H_{12} \\ 0 & H_{12}^* & H_{22} \end{bmatrix} \sim \begin{bmatrix} 0 & I_{\rho+\ell} & 0 \\ I_{\rho+\ell} & 0 & 0 \\ 0 & 0 & H_{22} \end{bmatrix} \\ &\sim \begin{bmatrix} -I_{\rho+\ell} & 0 & 0 \\ 0 & I_{\rho+\ell} & 0 \\ 0 & 0 & H_{22} \end{bmatrix}. \end{aligned} \quad (10)$$

Hence, by Sylvester's law of inertia [21], the matrix $\mathcal{N}(i\omega)$ has at least $(\rho+\ell)$ positive and at least $(\rho+\ell)$ negative eigenvalues for every $\omega \in \mathbb{R}$. Since $f_{\mathcal{N}}(\omega)$ is ordered non-increasingly, this implies that the function $f_{\mathcal{N}}$ can be partitioned into $f_{\mathcal{N}} =: [f_1^T, g_{\mathcal{N}}^T, f_3^T]^T$, where $f_1, f_3 : \mathbb{R} \rightarrow \mathbb{R}^{\rho+\ell}$ and $f_1(\omega)$ contains only positive entries whereas $f_3(\omega)$ contains only negative entries for every $\omega \in \mathbb{R}$. Thus, to determine the value of $\eta(\mathcal{N}(i\omega))$ in (6) it is sufficient to take into account the $k - 2(\rho + \ell) = n + m - \rho$ eigenvalues in $g_{\mathcal{N}}$ only. Thus, recalling (9), a system is dissipative if and only if all elements of $g_{\mathcal{N}}(\omega)$ are non-negative for all $\omega \in \mathbb{R}$. A plot of the function

$$g_{\mathcal{N}} : \mathbb{R} \rightarrow \mathbb{R}^{n+m-\rho}$$

is what we call a *spectral plot*. A typical example of a spectral plot is given in Fig. 1, where a system with $n = \rho$ and $m = 2$ is depicted.

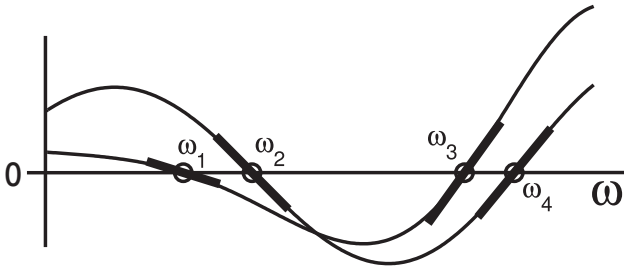


Fig. 1. Spectral plot of a system with two inputs; the thick bars show the slopes σ_j of the lines

In Fig. 1 the points where one of the lines becomes zero are marked by ω_j 's and also the slope/derivative of the lines at each ω_j are plotted as thick bars. We will denote the slope by σ_j in the following. Clearly, the system in Fig. 1 is not dissipative, because the curves are negative for some ω . However, it would be if the two bulges between ω_1 and

ω_4 could be "moved upwards". The method that will be developed in this paper will move ω_1 and ω_3 (and also ω_2 and ω_4) towards each other, which has the consequence that the negative bulge between them gets smaller and smaller and eventually vanishes, thereby reducing the number of roots ω_j , as depicted in Fig. 2.

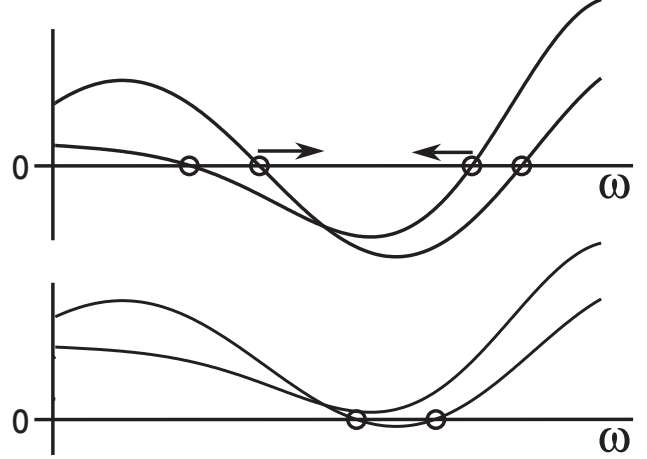


Fig. 2. Schematic sketch of system perturbation

Note, that in the spectral plot in Fig. 1 there exists a $\hat{\omega} \in \mathbb{R}$ at which both lines are different from 0; actually both lines are different from zero everywhere except at the points ω_j , for $j = 1, \dots, 4$. Thus, for such an $\hat{\omega}$ the matrix H_{22} in (10) is invertible, which implies that $\mathcal{N}(i\hat{\omega})$ is invertible. Using (5) this implies that \mathcal{N} is regular, which we will always assume in the following. In the same way, we obtain that for ω_j the matrix H_{22} in (10) is singular and thus we have $\text{rank}(\mathcal{N}(i\omega_j)) < k$, for $j = 1, \dots, 4$. This implies that $i\omega_j$ is a zero of \mathcal{N} . Analogously, the converse is true, namely, that the zeros of \mathcal{N} mark the points on the imaginary axis where one of the lines in the spectral plot changes from positive to negative (if $\sigma_j < 0$) or vice versa (if $\sigma_j > 0$). Thus, every ω_j accounts for a change of 2 in the value of $\eta(\mathcal{N}(i\omega))$, more precisely

$$\begin{aligned} \eta(\mathcal{N}(i(\omega_j - \varepsilon))) \\ = \eta(\mathcal{N}(i(\omega_j + \varepsilon))) + \begin{cases} 2 & \text{if } \sigma_j > 0 \\ -2 & \text{if } \sigma_j < 0 \end{cases} \end{aligned} \quad (11)$$

for a sufficiently small $\varepsilon > 0$. We distinguish two cases.

Definition 3. A dissipativity enforcement problem is called feasible, if condition (9) is violated in a bounded interval only. If, on the other hand, (9) is violated in an unbounded interval then the problem is called infeasible.

Note that in the special case of $\rho = n$, $E = I$ feasibility is implied by positive definiteness of $D^T + D$ (if the supply function is of the form (3)) or by $\|D\|_2 < 1$ (if the supply function is of the form (4)). When dealing with passivity, those two assumptions on D are standard, see, e.g., [1].

Example 4. An example of a feasible system is given in Fig. 1. Here we see that condition (9) is violated in the bounded

interval $[\omega_1, \omega_4]$ only. Thus, if all imaginary eigenvalues are moved into each other and away from the imaginary axis as discussed above (while not changing the value of $\eta(\mathcal{N}(i\omega))$ outside of $[\omega_1, \omega_4]$), we conclude dissipativity by Theorem 2.

An example of an infeasible system is given in Fig. 3, where condition (9) is violated in the unbounded interval $[\omega_6, \infty)$.

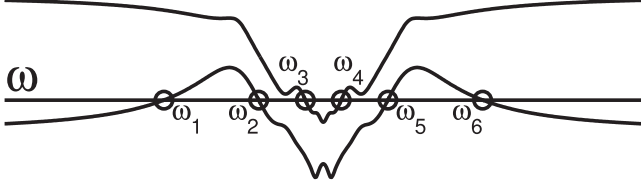


Fig. 3. Spectral plot of an infeasible system with two inputs; without slopes

Thus, even if we move all purely imaginary zeros away from the imaginary axis (i.e., by merging ω_1 with ω_2 , ω_3 with ω_4 , and ω_5 with ω_6) condition (9) is still violated, implying a non-dissipative system.

From the above it is clear that the two cases can be distinguished numerically in the following way. First one computes the value of $a := \eta(\mathcal{N}(i\hat{\omega}))$ for some fixed $\hat{\omega} \notin \zeta(\mathcal{N})$. Then one counts the number of positive slopes $\sigma_j > 0$ (say a_1) and the number of negative slopes $\sigma_j < 0$ (say a_2) for all $\omega_j > \hat{\omega}$. Analogously, for all $\omega_j < \hat{\omega}$, one computes the number of positive slopes $\sigma_j > 0$ (say b_1) and the number of negative slopes $\sigma_j < 0$ (say b_2). Then, using (11), we see that if

$$\begin{aligned} n + m - \rho &= \eta(\mathcal{N}(i\hat{\omega})) + 2(a_1 - a_2) \\ &= \eta(\mathcal{N}(i\hat{\omega})) + 2(b_1 - b_2), \end{aligned} \quad (12)$$

we are in the feasible case, otherwise we are in the infeasible case. In the following we will only consider the feasible case.

Next we want to compute the previously announced perturbation that moves the imaginary eigenvalues closer together or even completely off the imaginary axis. To this end we allow perturbations in M , i.e., in A, B, C, D, Q, R , and S . Allowing all these matrices to change results in smaller perturbations compared to the situation when just C may change as was the case in earlier methods [1], [13], [14], [15] of this kind. On the other hand, we do not allow perturbations in N (that is E), although this should allow even smaller perturbations in some cases, because that could easily change the rank of N completely altering the spectral plots for $\omega \rightarrow \infty$. More research is needed to gain enough insight so that also E can be perturbed. First steps in this direction were undertaken in [22].

III. PERTURBATION THEORY

Let $\lambda E - A \in \mathbb{R}[\lambda]^{n,n}$ be a regular matrix polynomial and let $\lambda_0 \in \zeta(\lambda E - A)$ be a zero, i.e., assume that $\text{rank}(\lambda_0 E - A) < \text{rank}_{\mathbb{R}(\lambda)}(\lambda E - A) = n$. This shows that the matrix $(\lambda_0 E - A) \in \mathbb{C}^{n,n}$ is singular and thus there exist vectors $u_0, v_0 \in \mathbb{C}^n \setminus \{0\}$ such that

$$(\lambda_0 E - A)v_0 = 0 \quad \text{and} \quad u_0^*(\lambda_0 E - A) = 0.$$

In this case (i.e., only if $\lambda E - A$ is regular) we call v_0 (and u_0) a right (and left) eigenvector associated with the eigenvalue

λ_0 . We note again that for regular matrix polynomials the notions of "eigenvalue" and "zero" coincide.

Theorem 5. [23, p.183] Let $\lambda_0 \in \mathbb{C}$ be a simple finite eigenvalue of the regular pencil $\lambda E - A \in \mathbb{R}[\lambda]^{n,n}$ with right eigenvector v_0 and left eigenvector u_0 . Then there exists a function $\phi : \mathbb{R}^{n,n} \rightarrow \mathbb{C}$ which fulfills $\phi(\Lambda) = \mathcal{O}(\|\Lambda\|^2)$ for $\Lambda \rightarrow 0$ such that

$$\tilde{\lambda}_0 = \lambda_0 + \frac{u_0^* \Lambda v_0}{u_0^* E v_0} + \phi(\Lambda) \quad (13)$$

is an eigenvalue of the perturbed pencil $\lambda E - (A + \Lambda)$.

Assume that $\lambda_0 = i\omega_0$ is a purely imaginary eigenvalue of the regular and para-Hermitian pencil $\lambda N - M \in \mathbb{R}[\lambda]_1^{k,k}$ associated with the eigenvector x_0 , i.e., assume that we have $0 = i\omega_0 N x_0 - M x_0$. Taking the conjugate transpose of this equation and using that N is skew-symmetric and that M is symmetric (since $\lambda N - M$ is para-Hermitian) shows that

$$0 = x_0^* N^* \overline{i\omega_0} - x_0^* M^* = -i\omega_0 x_0^* (-N) - x_0^* M^*,$$

i.e., that x_0 is also a left eigenvector associated with $i\omega_0$. We conclude that for regular and para-Hermitian matrix polynomials the left and right eigenvectors of purely imaginary eigenvalues coincide.

With this, we can use Theorem 5 to determine the slopes of the lines in the spectral plot at the ω_j 's. Therefore, let x_j be a generalized eigenvector of the pencil \mathcal{N} (with \mathcal{N} given by (8)) associated with the purely imaginary eigenvalue ω_j . This implies that x_j is a standard (left and right) eigenvector of the constant matrix $\mathcal{N}(i\omega_j)$ associated with the eigenvalue 0, i.e., that $\mathcal{N}(i\omega_j)x_j = 0$ and $x_j^* \mathcal{N}(i\omega_j) = 0$. To find out what happens to this eigenvalue in a small neighborhood of ω_j we observe that the matrix $\mathcal{N}(i(\omega + \Delta\omega)) = \mathcal{N}(i\omega) + i\Delta\omega N$ has (due to (13) with $E = I$, $\lambda_j = 0$, $\Lambda = i\Delta\omega N$) an eigenvalue at

$$\tilde{\lambda} = 0 + i\Delta\omega \frac{x_j^* N x_j}{x_j^* x_j} + \phi(i\Delta\omega N)$$

which means that for normalized eigenvectors $\|x_j\|_2 = 1$ the derivative is given by

$$\sigma_j = \lim_{\Delta\omega \rightarrow 0} \frac{\tilde{\lambda} - 0}{\Delta\omega} = i x_j^* N x_j. \quad (14)$$

Note that σ_j is real, because N is skew-Hermitian. Let us remark that the slopes used in [1] are fractions of the terms used here and denominators involving inverse matrices. The cost to evaluate these fractions led to the advise to not use slopes at all [14].

If we apply Theorem 5 to the regular and para-Hermitian matrix pencil $\lambda N - (M + \Delta) \in \mathbb{R}[\lambda]_1^{k,k}$ (with symmetric perturbation Δ) and the purely imaginary eigenvalue ω_j with eigenvector x_j then, since in this case the left and right eigenvectors coincide, equation (13) reads (neglecting higher order terms) $i\tilde{\omega}_j \approx i\omega_j + \frac{x_j^* \Delta x_j}{x_j^* N x_j}$, which by multiplication with $x_j^* N x_j$ and the use of (14) leads to

$$\sigma_j(\tilde{\omega}_j - \omega_j) \approx x_j^* \Delta x_j. \quad (15)$$

Note that both sides of (15) are real numbers.

Looking at (15) we see that the perturbation matrix Δ is what we actually want to compute, i.e., Δ is the indeterminate. To transform the formula (15) into a linear equation in the usual form, we need the following definitions. Let $W \in \mathbb{C}^{n,m}$ be a matrix with columns $w_1, \dots, w_m \in \mathbb{C}^n$. Then we define the *vec operator* $\text{vec} : \mathbb{C}^{n,m} \rightarrow \mathbb{C}^{nm}$ by

$$\text{vec}(W) := \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

If additionally $V \in \mathbb{C}^{p,q}$ is a matrix with entries $v_{i,j}$ then we define the *Kronecker product* $\otimes : \mathbb{C}^{p,q} \times \mathbb{C}^{n,m} \rightarrow \mathbb{C}^{pn,qm}$ through

$$V \otimes W := \begin{bmatrix} v_{1,1}W & \dots & v_{1,q}W \\ \vdots & & \vdots \\ v_{p,1}W & \dots & v_{p,q}W \end{bmatrix}.$$

Combining these two definitions one can show by basic manipulations that

$$x^* \Delta y = (y^T \otimes x^*) \text{vec}(\Delta) \quad (16)$$

which is a linear equation in $\text{vec}(\Delta)$.

IV. DERIVATION OF EQUATIONS FOR DISSIPATIVITY ENFORCEMENT

For our method we assume that \mathcal{N} as in Theorem 2 is regular and that the problem is feasible according to Definition 3. From condition (12) we already deduced that for feasible problems the number of purely imaginary eigenvalues of \mathcal{N} is even, say 2θ . We denote these eigenvalues as $\omega_1, \dots, \omega_{2\theta} \in i\mathbb{R}$ and assume that $\omega_1 < \dots < \omega_{2\theta}$. The corresponding eigenvectors are called $x_1, \dots, x_{2\theta} \in \mathbb{C}^k \setminus \{0\}$ and we assume w.l.o.g. that

$$\|x_i\|_2 = 1 \quad \text{for all } i = 1, \dots, 2\theta.$$

Since $N, M \in \mathbb{R}^{k,k}$ are real matrices we have that $\omega_i = -\omega_{2\theta+1-i}$ which, using the notation

$$\omega_{-i} := \omega_{2\theta+1-i}, \quad x_{-i} := x_{2\theta+1-i}, \quad (17)$$

can be rewritten as

$$\omega_{-i} = -\omega_i, \quad (18)$$

for $i = 1, \dots, \theta$. This implies that $\omega_{-i} N \overline{x_i} = -\omega_i N \overline{x_i} = \overline{\omega_i N x_i} = \overline{M x_i} = M \overline{x_i}$, from which one can deduce that $\overline{x_i}$ is an eigenvector of the eigenvalue ω_{-i} for $i = 1, \dots, \theta$. Thus, w.l.o.g. one can assume that

$$x_{-i} = \overline{x_i}, \quad (19)$$

since otherwise this could be achieved by multiplying with a complex number of absolute value 1. Using (14) we observe that with this we have

$$\begin{aligned} \sigma_{-i} &= ix_{-i}^* N x_{-i} = i \overline{x_i}^* N \overline{x_i} = -i (x_i^* N^* x_i)^* \\ &= -\sigma_i. \end{aligned} \quad (20)$$

Our goal in this section is to compute a symmetric perturbation of M , which we call $\Delta = \Delta^T \in \mathbb{R}^{k,k}$, such that the

eigenvalue ω_i in the unperturbed pencil $\lambda N - M$ moves to the eigenvalue $\tilde{\omega}_i$ in the perturbed pencil $\lambda N - (M + \Delta)$, for $i = 1, \dots, 2\theta$. For the moment, we will assume that we know these positions $\tilde{\omega}_i$ and we will later describe different methods to determine them in Subsection VI-A. Then (15) and (16) show that $\Delta = \Delta^T$ has to satisfy

$$\sigma_i(\tilde{\omega}_i - \omega_i) \approx (x_i^T \otimes x_i^*) \text{vec}(\Delta), \quad (21)$$

for $i = 1, \dots, 2\theta$. Using (18), (19), (20), and making the reasonable assumption that $\tilde{\omega}_{-i} = -\tilde{\omega}_i$ we see that for $i = 1, \dots, 2\theta$ we have

$$\begin{aligned} \sigma_{-i}(\tilde{\omega}_{-i} - \omega_{-i}) &= (x_{-i}^T \otimes x_{-i}^*) \text{vec}(\Delta) \\ \Leftrightarrow \sigma_i(\tilde{\omega}_i - \omega_i) &= (\overline{x_i^T} \otimes \overline{x_i^*}) \text{vec}(\Delta) \\ \Leftrightarrow \sigma_i(\tilde{\omega}_i - \omega_i) &= \overline{(x_i^T \otimes x_i^*) \text{vec}(\Delta)}. \end{aligned}$$

Thus, we can rewrite (21) as

$$\begin{aligned} \sigma_i(\tilde{\omega}_i - \omega_i) &\approx (x_i^T \otimes x_i^*) \text{vec}(\Delta), \\ \sigma_i(\tilde{\omega}_i - \omega_i) &\approx \overline{(x_i^T \otimes x_i^*) \text{vec}(\Delta)}, \end{aligned}$$

for $i = 1, \dots, \theta$. Of course, we would like to have a system of equations, in which all coefficients are real numbers.

Lemma 6. *Let $\alpha \in \mathbb{C}$ and $v \in \mathbb{C}^{1,n}$. Then the system of equations in the unknown $\delta \in \mathbb{R}^n$ given by*

$$\begin{aligned} \alpha &= v\delta \\ \overline{\alpha} &= \overline{v}\delta \end{aligned} \quad (22)$$

is equivalent to the system of equations

$$\begin{aligned} \text{Re}(\alpha) &= \text{Re}(v)\delta \\ \text{Im}(\alpha) &= \text{Im}(v)\delta. \end{aligned} \quad (23)$$

Proof: The result follows by basic manipulation. \blacksquare

Since $\sigma_i(\tilde{\omega}_i - \omega_i) \in \mathbb{R}$ is a real number, Lemma 6 shows that (21) is equivalent to

$$\begin{bmatrix} \sigma_1(\tilde{\omega}_1 - \omega_1) \\ \vdots \\ \sigma_\theta(\tilde{\omega}_\theta - \omega_\theta) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \approx \begin{bmatrix} \text{Re}(x_1^T \otimes x_1^*) \\ \vdots \\ \text{Re}(x_\theta^T \otimes x_\theta^*) \\ \text{Im}(x_1^T \otimes x_1^*) \\ \vdots \\ \text{Im}(x_\theta^T \otimes x_\theta^*) \end{bmatrix} \text{vec}(\Delta).$$

Since the entries of the symmetric matrix Δ are real numbers, we see that for every $x \in \mathbb{C}^k$ we have

$$\begin{aligned} \text{Im}(x^T \otimes x^*) \text{vec}(\Delta) &= \text{Im}((x^T \otimes x^*) \text{vec}(\Delta)) \\ &= \text{Im}(x^* \Delta x) = 0. \end{aligned}$$

This implies, that (21) is also equivalent to

$$\begin{bmatrix} \sigma_1(\tilde{\omega}_1 - \omega_1) \\ \vdots \\ \sigma_\theta(\tilde{\omega}_\theta - \omega_\theta) \end{bmatrix} \approx \begin{bmatrix} \text{Re}(x_1^T \otimes x_1^*) \\ \vdots \\ \text{Re}(x_\theta^T \otimes x_\theta^*) \end{bmatrix} \text{vec}(\Delta)$$

which, with the notation

$$\begin{aligned} \omega &:= \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_\theta \end{bmatrix}, & \Sigma &:= \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_\theta \end{bmatrix}, \\ \tilde{\omega} &:= \begin{bmatrix} \tilde{\omega}_1 \\ \vdots \\ \tilde{\omega}_\theta \end{bmatrix}, & X &:= \begin{bmatrix} \operatorname{Re}(x_1^T \otimes x_1^*) \\ \vdots \\ \operatorname{Re}(x_\theta^T \otimes x_\theta^*) \end{bmatrix}, \end{aligned}$$

can be written as

$$\Sigma(\tilde{\omega} - \omega) \approx X \operatorname{vec}(\Delta). \quad (24)$$

Looking at the matrix M in Theorem 2 we conclude that we do not want to allow for arbitrary symmetric perturbations Δ , especially we do not want to perturb the zero blocks in the matrix M . We follow the approach that we only allow for perturbations from a linear subspace of the symmetric matrices. Therefore, let $K \in \mathbb{N}$ and let $\Delta_1, \dots, \Delta_K \in \mathbb{R}^{k,k}$ be a basis of that linear subspace of the symmetric matrices, which we want to allow. Since we want $\Delta \in \operatorname{span}(\Delta_1, \dots, \Delta_K)$ we represent Δ in the form

$$\Delta = \sum_{i=1}^K \Delta_i \delta_i. \quad (25)$$

Introduce

$$\delta := \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_K \end{bmatrix}, \quad \text{and} \quad \mathcal{D} := [\operatorname{vec}(\Delta_1) \quad \dots \quad \operatorname{vec}(\Delta_K)]$$

and observe that with this we have

$$\operatorname{vec}(\Delta) = \operatorname{vec}\left(\sum_{i=1}^K \Delta_i \delta_i\right) = \sum_{i=1}^K \operatorname{vec}(\Delta_i) \delta_i = \mathcal{D} \delta. \quad (26)$$

Thus, equation (24) becomes

$$\Sigma(\tilde{\omega} - \omega) \approx X \mathcal{D} \delta. \quad (27)$$

We remark that the matrix product $X \mathcal{D}$ can be formed directly, i.e., without explicitly computing X and \mathcal{D} . Indeed, using that Δ_j is real, we have for the (i, j) -element of $X \mathcal{D}$

$$\begin{aligned} (X \mathcal{D})_{ij} &= \operatorname{Re}(x_i^T \otimes x_i^*) \operatorname{vec}(\Delta_j) \\ &= \operatorname{Re}(x_i^* \Delta_j x_i) \end{aligned}$$

for all $i = 1, \dots, \theta$, $j = 1, \dots, K$. Proceeding like this seems advisable whenever $K \ll n^2$.

V. DERIVATION OF EQUATIONS FOR STABILIZATION

In many applications, in which one deals with systems of the form (1), one is also interested in the stability of $\lambda E - A$. Here, we call a matrix pencil $\lambda E - A$ *stable* if all its finite zeros are in the open left half plane.

Remark 7. *There are stronger notions of stability. For example, one can additionally require impulse controllability, i.e., that*

$$\operatorname{rank}[E, AS_\infty, B] = n, \quad (28)$$

where the columns of the matrix S_∞ form an orthonormal basis of $\operatorname{kernel}(E)$, see [24]. If this notion is strived for, condition (28) can easily be checked numerically.

For numerical reasons we only consider closed stability regions of the form

$$\mathbb{C}_\xi := \{z \in \mathbb{C} \mid \operatorname{Re}(z) \leq \xi\},$$

where $\xi \in \mathbb{R} \cup \{\infty\}$. Choosing $\xi < 0$ guarantees that all zeros are bounded away from the imaginary axis.

If a non-trivial stability region (i.e., $\xi < \infty$) is desired, we additionally have to assume that the pencil $\lambda E - A \in \mathbb{R}[\lambda]^{\rho,n}$ is regular (which among others implies $\rho = n$) so that we can talk about eigenvalues and eigenvectors. The aim is to find a perturbation Λ of A such that $\lambda E - (A + \Lambda)$ is stable. More precisely, denoting the unstable eigenvalues of $\lambda E - A$ by μ_i , $i = 1, 2, \dots$ (i.e., $\operatorname{Re}(\mu_i) > \xi$) and the corresponding eigenvalues of $\lambda E - (A + \Lambda)$ by $\tilde{\mu}_i$, we are looking for a perturbation such that

$$\operatorname{Re}(\tilde{\mu}_i) = \xi. \quad (29)$$

Note, that we only prescribe the real part of the moved eigenvalues, while leaving the imaginary part free. A naive approach would have been to set $\tilde{\mu}_i = \xi + \operatorname{Im}(\mu_i)$, but placing less conditions usually results in a smaller perturbation Λ .

We remark that the perturbation of A will in general move all eigenvalues, not just the unstable ones. So it can not be excluded that some previously stable eigenvalue will become unstable. This situation will be detected and dealt with at a later stage.

We divide the unstable eigenvalues into two classes: i) those that are real and ii) those that form complex conjugate pairs. Let us say that there are β complex conjugate pairs and γ real unstable eigenvalues and that $\mu_1, \dots, \mu_{2\beta+\gamma}$ are ordered by their imaginary part. Then $\mu_i = \overline{\mu_{2\beta+\gamma+1-i}}$ for $i = 1, \dots, 2\beta + \gamma$ and $\operatorname{Im}(\mu_i) \geq 0$ for $i = 1, \dots, \beta + \gamma$. Introducing the notation

$$\mu_{-i} := \mu_{2\beta+\gamma+1-i}, \quad u_{-i} := u_{2\beta+\gamma+1-i}, \quad v_{-i} := v_{2\beta+\gamma+1-i}$$

in the style of (17) this means that

$$\mu_i = \overline{\mu_{-i}},$$

for $i = 1, \dots, 2\beta + \gamma$ and we restrict the moved eigenvalues $\tilde{\gamma}_i$ to satisfy analogous relations, i.e.,

$$\tilde{\mu}_i = \overline{\tilde{\mu}_{-i}} \quad \text{and} \quad \tilde{\mu}_{\beta+1}, \dots, \tilde{\mu}_{\beta+\gamma} \in \mathbb{R}.$$

For $i = 1, \dots, 2\beta + \gamma$ we denote right and left eigenvectors associated with μ_i by $v_i \in \mathbb{C}^n$ and $u_i \in \mathbb{C}^n$, respectively, chosen such that

$$\begin{aligned} v_i &= \overline{v_{-i}}, \quad u_i = \overline{u_{-i}} \quad \text{for } i = 1, \dots, \beta; \\ v_i, u_i &\in \mathbb{R}^n \quad \text{for } i = \beta + 1, \dots, \beta + \gamma; \\ u_i^* E v_i &\in \mathbb{R} \quad \text{for } i = 1, \dots, 2\beta + \gamma. \end{aligned}$$

Then, by Theorem 5, Λ has to satisfy the conditions

$$u_i^* E v_i (\tilde{\mu}_i - \mu_i) \approx u_i^* \Lambda v_i,$$

for $i = 1, \dots, 2\beta + \gamma$. For $i = \beta + 1, \dots, \beta + \gamma$ these conditions are real. For the complex conditions we can separate real and

imaginary parts and, invoking Lemma 6, obtain the alternative set of conditions

$$u_i^* E v_i \cdot \operatorname{Re}(\tilde{\mu}_i - \mu_i) \approx \operatorname{Re}(u_i^* \Lambda v_i), \quad (30)$$

for $i = 1, \dots, \beta + \gamma$ and

$$u_i^* E v_i \cdot \operatorname{Im}(\tilde{\mu}_i - \mu_i) \approx \operatorname{Im}(u_i^* \Lambda v_i), \quad (31)$$

for $i = 1, \dots, \beta$. Since (29) only specifies the real part of $\tilde{\mu}_i$, the conditions (31) are not necessary, and only the conditions (30) remain.

So far, we have obtained conditions on the matrix Λ . However, we rather want conditions on the symmetric perturbation Δ . We see from (8) that both perturbations are related via

$$\Delta = \begin{bmatrix} \times & \times & \times & \Lambda & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \Lambda^T & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}, \quad (32)$$

where \times denotes block matrices of appropriate size. Defining

$$\begin{aligned} \check{u}_i &:= [u_i^T \ 0 \ 0 \ 0 \ 0]^T, \\ \check{v}_i &:= [0 \ 0 \ 0 \ v_i^T \ 0]^T, \end{aligned} \quad (33)$$

analogously to the partitioning in (8) (or (32)) we see that with (16) and (26) we have

$$u_i^* \Lambda v_i = \check{u}_i^* \Delta \check{v}_i = (\check{v}_i^T \otimes \check{u}_i^*) \operatorname{vec}(\Delta) = (\check{v}_i^T \otimes \check{u}_i^*) \mathcal{D} \delta.$$

Thus, we can rewrite (30) as

$$u_i^* E v_i \cdot \operatorname{Re}(\tilde{\mu}_i - \mu_i) \approx \operatorname{Re}(\check{v}_i^T \otimes \check{u}_i^*) \mathcal{D} \delta, \quad (34)$$

for $i = 1, \dots, \beta + \gamma$. Finally, we introduce

$$\begin{aligned} \mu &:= \begin{bmatrix} \operatorname{Re}(\mu_1) \\ \vdots \\ \operatorname{Re}(\mu_{\beta+\gamma}) \end{bmatrix}, & \tilde{\mu} &:= \begin{bmatrix} \xi \\ \vdots \\ \xi \end{bmatrix} \\ Y &:= \begin{bmatrix} \operatorname{Re}(\check{v}_1^T \otimes \check{u}_1^*) \\ \vdots \\ \operatorname{Re}(\check{v}_{\beta+\gamma}^T \otimes \check{u}_{\beta+\gamma}^*) \end{bmatrix}, \end{aligned}$$

and

$$T := \operatorname{diag}(u_1^* E v_1, \dots, u_{\beta+\gamma}^* E v_{\beta+\gamma}).$$

Note that with $u_i^* E v_i$ also T is real. Then we can rewrite (34) in matrix form as

$$T(\tilde{\mu} - \mu) \approx Y \mathcal{D} \delta. \quad (35)$$

VI. THE METHOD

In the two previous sections we have derived conditions on the perturbation Δ such that the eigenvalues that prevent dissipativity or stability are moved to harmless positions. What remains is determining the smallest perturbation Δ (in the sense that $\|\delta\|_2$ in (26) is minimized) satisfying the two conditions (27) and (35). This is a standard least squares problem with solution

$$\delta = Q_1 R_1^{-T} \begin{bmatrix} \Sigma(\tilde{\omega} - \omega) \\ T(\tilde{\mu} - \mu) \end{bmatrix}, \quad (36)$$

where $\mathcal{D}^T[X^T, Y^T] = Q_1 R_1$ denotes a skinny QR factorization [25, p. 230] and $\Sigma, \tilde{\omega}, \omega, T, \tilde{\mu}, \mu, X, Y$, and \mathcal{D} are as in (27) and (35). Once δ is known, Δ can be computed and M (i.e. A, B, C, D, Q, R, S) can be updated.

We have the following algorithm.

Algorithm 8. (Dissipativity Enforcement)

Input: The matrices of the system (1) and the matrices of the supply (2) such that the resulting pencil (8) is regular and the problem is feasible, i.e., condition (7) is fulfilled.

A basis of the perturbation (25), such that each basis element Δ_i only has non-zero entries in those places where the matrix M in (8) has a non-zero block.

An integer describing the maximum number of iterations, MAXITER. The stability parameter from Section V, which is $\xi \in \mathbb{R} \cup \{\infty\}$. $\lambda E - A$ is additionally required to be regular, if $\xi \neq \infty$.

Output: An error if the problem is infeasible, otherwise a dissipative system or a message that dissipativity was not possible.

Step 1: Check if the problem is feasible or infeasible via condition (12). Return with an error if infeasible. Otherwise, initialize the iteration counter ITER := 0.

Step 2: Form the para-Hermitian pencil $\lambda N - M$ as in (8).

Step 3: Compute the purely imaginary eigenvalues and associated eigenvectors of the pencil $\lambda N - M$.

Step 4: If $\lambda E - A$ is regular, compute the eigenvalues and associated eigenvectors of $\lambda E - A$, which are not in the stability region \mathbb{C}_ξ , see Section V.

Step 5: If (8) has no more purely imaginary eigenvalues and no eigenvalues outside the stability region, goto 10. Otherwise, if ITER > MAXITER, goto 10 and return with a message, indicating that the dissipativity failed.

Step 6: Choose $\tilde{\omega}$, for example, as described in the following subsection.

Step 7: Assemble the matrices in (27) and (35) and compute δ by (36).

Step 8: Use (25) to compute the perturbation Δ and update $M \rightarrow M + \Delta$.

Step 9: Increase ITER and goto 3.

Step 10: Break up the actual M according to the block structure from (8) to obtain the perturbed system and supply matrices A, B, C, D, Q, S , and R .

The complexity of Algorithm 8 is essentially determined by the eigenvalue computations in Steps 3 & 4 and the QR-factorization in Step 6. To analyze the complexity assume that in (1) we have $\rho = n$ and that the number of inputs and outputs $m + \ell$ is small compared to n . This implies that the eigenvalue computations in Steps 3 and 4 take $\mathcal{O}(n^3)$ flops.

The number of basis elements, in the case where n is dominating, can be assumed to be $K = \mathcal{O}(n^2)$. At the same time, the number of (purely imaginary) eigenvalues of the pencil (8) can be bounded by $\mathcal{O}(n)$ and the number of eigenvalues of $\lambda E - A$ as in (1) is smaller than n . This implies that the total number of relevant eigenvalues $p := \theta + 2\beta + \gamma$

is at most in the order of $\mathcal{O}(n)$ although it will usually be much smaller. Since a skinny QR factorization of the K -by- p matrix $\mathcal{D}^T[X^T, Y^T]$ takes $\mathcal{O}(p^2K)$ operations we conclude that Step 6, and thus one iteration of Algorithm 8, has a worst case runtime of

$$\mathcal{O}(n^4),$$

although in practice it will rather be $\mathcal{O}(n^3)$, if the number purely imaginary eigenvalues of (8) and the number of unstable eigenvalues of $\lambda E - A$ as in (1) is bounded.

Note that the updated system is neither guaranteed to be dissipative nor stable, because a) we omitted higher order terms during the derivation of the method and b) non-imaginary eigenvalues $\lambda N - M$ (or stable eigenvalues of $\lambda E - A$ respectively) could have moved on the imaginary axis (outside the stability region). Thus, the perturbed system needs to be checked for dissipativity and stability again. If either is violated, then a second, a third, \dots perturbation has to be computed in an iterative fashion, until the obtained system is both dissipative and stable.

Usually this iteration finishes within a few steps only, see our numerical examples in Section VII. However, it is not guaranteed that the iteration ever completes successfully. In fact, there are examples where the algorithm will iterate forever.

Example 9. Consider the system

$$E = I_2, \quad A = -4I_2, \quad B = \text{diag}[5, 3], \quad C = I_2, \quad D = 0$$

with supply function

$$Q = -I_2, \quad S = 0, \quad R = I_2,$$

and the single perturbation basis matrix

$$\Delta_1 = -2(e_1e_9^T + e_9e_1^T) + 2(e_2e_{10}^T + e_{10}e_2^T) \in \mathbb{R}^{10,10}.$$

Setting $\lambda N - M$ as in (8), its finite spectrum is $\{\pm 3i, \pm\sqrt{7}\}$. We have $\omega_1 = -3$, $x_1 = [1, 4 + 3i, -4 - 3i, 4 + 3i, 5]^T \otimes [1, 0]^T$, $\sigma_1 = -6$, $XD = -20$. Choosing $\tilde{\omega}_1 = \frac{1}{3}$ (e.g., by (38) with $\tau = \frac{5}{9}$) we get $\delta^{(1)} = 1$. Applying the perturbation, the finite spectrum of the perturbed pencil $\lambda N - (M + \Delta_1)$ is now given by $\{\pm\sqrt{7}, \pm 3i\}$. Since there are still imaginary eigenvalues, the perturbed system is still not dissipative and a second iteration of Algorithm 8 is necessary. This time $\delta^{(2)}$ comes out to be -1 , thus annihilating the first perturbation $(M + \Delta_1) - \Delta_1$. So, the iteration will toggle between M and $M + \Delta_1$ forever.

Note that in this example there is exactly one choice of δ achieving dissipativity. Indeed, the finite spectrum of $\lambda N - (M + \delta\Delta_1)$ is given by $\lambda_{1,2} = \pm\sqrt{(1-2\delta)(2\delta-9)}$ and $\lambda_{3,4} = \pm\sqrt{(1-2\delta)(2\delta+7)}$. So, for the value of $\delta = \frac{1}{2}$ the pencil $\lambda N - (M + \frac{1}{2}\Delta_1)$ has four eigenvalues at zero. A main reason for this failure is the extremely limited set of allowable perturbations. The effect here is that a perturbation that moves a pair of imaginary eigenvalues off the imaginary axis, will also move the other pair of eigenvalues onto the imaginary axis, and vice versa. A remedy is to allow more perturbations. If, for example, $\Delta_2 = -2(e_2e_{10}^T + e_{10}e_2^T)$ is added as a further perturbation basis matrix then the first iteration of

Algorithm 8 will not change and still result in $\delta^{(1)} = [1, 0]^T$. But in the second iteration we get $\delta^{(2)} = [-\frac{1}{2}, \frac{1}{2}]$, yielding $\lambda N - (M + \frac{1}{2}\Delta_1 + \frac{1}{2}\Delta_2)$ with two non-imaginary eigenvalues and two eigenvalues at zero.

More than anything, Example 9 shows that a sufficiently broad set of perturbation basis matrices is crucial for the success of the method.

A different approach to the problem that moving imaginary eigenvalues off the imaginary axis may cause previously non-imaginary eigenvalues to move on the axis is employed in [16]: There the perturbations are restricted to the invariant subspace corresponding to the imaginary eigenvalues thereby making sure that the non-imaginary eigenvalues will not move at all. This, however, yields a dense perturbation Δ that does not comply with the zero pattern of M ; and just setting those elements to zero would again admit the non-imaginary eigenvalues to move. We prefer to choose Δ as discussed above and note that our method is heuristic.

It remains to discuss how to choose the values $\tilde{\omega}_i$ and the matrices Δ_i . This is done in the following subsections together with a short discussion of how the computation of imaginary eigenvalues at Step 3 of Algorithm 8 can be carried out.

A. Choice of $\tilde{\omega}$

In this subsection we discuss different ways to choose the $\tilde{\omega}_i$ for $i = 1, \dots, \theta$, which are introduced in Section IV. Since the imaginary eigenvalues are to be moved towards each other, a basic requirement is that $\tilde{\omega}_i > \omega_i$ if $\sigma_i < 0$ and that $\tilde{\omega}_i < \omega_i$ whenever $\sigma_i > 0$. Hence, a straight forward choice for $\tilde{\omega}_i$ is given by

$$\tilde{\omega}_i := \omega_i - \tau \cdot \text{sign}(\sigma_j). \quad (37)$$

Here, as well as for the following choices below, τ denotes a positive constant that is an additional input to Algorithm 8.

This simple strategy works surprisingly well in practice, but has problems if dissipativity is violated in intervals of greatly varying lengths. In that case it makes sense to choose the distance $|\tilde{\omega}_i - \omega_i|$ relative to the distance of ω_i to its neighbors $\omega_{i\pm 1}$, i.e.,

$$\tilde{\omega}_i := \omega_i - \tau \cdot \begin{cases} (\omega_i - \omega_{i-1}), & \text{if } \sigma_i > 0 \\ (\omega_i - \omega_{i+1}), & \text{if } \sigma_i < 0 \end{cases}. \quad (38)$$

Note, that this choice (employed in [1]) is also well-defined for the border cases $i \in \{1, \theta\}$, because we know that $\sigma_1 < 0$, $\sigma_\theta > 0$ (since we assume feasibility).

The choice (38) works well as long as only one curve in the spectral plot falls below zero. But, if two or more curves fall below zero in one interval, then some zeros ω_i are moved only a very small distance (e.g., the zeros ω_1 and ω_4 in Fig. 1). This motivates the following refined choice (introduced in [13]): choose the distance $|\tilde{\omega}_i - \omega_i|$ relative to the distance of ω_i to its neighbor of opposite sign of slope, i.e.,

$$\tilde{\omega}_i := \omega_i - \tau \cdot \begin{cases} (\omega_i - \omega_{i^*}), & \text{if } \sigma_i > 0 \\ (\omega_i - \omega_{i^{**}}), & \text{if } \sigma_i < 0 \end{cases}, \quad (39)$$

where

$$\begin{aligned} i^* &= \max\{j \in \{1, \dots, i-1\} : \sigma_i \cdot \sigma_j < 0\}, \\ i^{**} &= \min\{j \in \{i+1, \dots, \theta\} : \sigma_i \cdot \sigma_j < 0\}. \end{aligned}$$

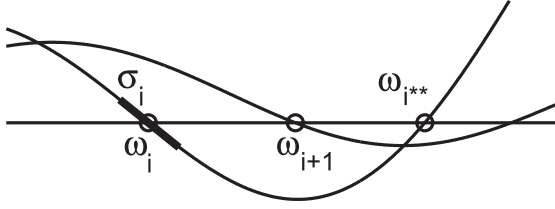


Fig. 4. Illustration of the local parameters from which the new position $\tilde{\omega}_i$ can be computed

B. Choice of Δ_i

Let us offer some thoughts on how to choose the perturbation basis matrices Δ_i . First of all, every Δ_i should be symmetric, since M is. Hence, with $M = [m_{ij}]_{i,j=1}^k$, a reasonable perturbation basis is given by

$$\{\mathcal{E}_{ij} : m_{ij} \text{ is allowed to change}\} \quad (40)$$

with

$$\mathcal{E}_{ij} = \begin{cases} \frac{1}{\sqrt{2}}(e_i e_j^T + e_j e_i^T), & i \neq j, \\ e_i e_i^T, & i = j, \end{cases}$$

where e_i denotes the i -th unit vector. Because the perturbation to M needs to be interpretable as perturbations to the system matrices, m_{ij} is allowed to change, only if it corresponds to an element of (some of) the matrices A, B, C, D, E, Q, R, S in (8). To decide which of those matrices should be allowed to change, a rule of thumb is that everything that is derived from measurements, linearizations or other potentially inaccurate computations should be allowed to change whereas tangible matrices like zero or identity blocks (e.g., Q, R, S in the cases (3) or (4)) should stay constant.

Another reasonable rule would be to allow only nonzero elements m_{ij} to change, which implies that the perturbation Δ respects the sparsity pattern of M . So, if n is large and the matrices are stored in sparse format, then the perturbed system does not need more storage than the original one. A further consequence is that structures that result in certain sparsity patterns (like higher order systems, cf. Section VIII, or coordinated systems [26]) are preserved by our dissipativity enforcement method.

Note that the choice (40) is easily adaptable to symmetric systems (i.e., E, A, D symmetric, $B = C^T$) such that also the perturbed system is symmetric.

Note that for the choice (40) we have $\|\delta\|_2 = \|\Delta\|_F$. Since $\|\delta\|_2$ is minimized by (36), also the Frobenius norm of Δ is minimized. Furthermore, a scaling of the matrices Δ_i may be used to minimize a weighted Frobenius norm. For example, scaling the basis matrices in (40) by $|m_{ij}|$, i.e., employing the basis

$$\{|m_{ij}| \mathcal{E}_{ij} : m_{ij} \neq 0 \text{ and } m_{ij} \text{ is allowed to change}\},$$

allows larger perturbations in large entries and smaller perturbations in small entries of M .

C. Computation of imaginary eigenvalues

In step 3 of Algorithm 8 we have to compute all imaginary eigenvalues of a para-Hermitian pencil $\lambda N - M \in \mathbb{R}[\lambda]^{k,k}$ and the corresponding eigenvectors. The approach to compute all eigenpairs (e.g., using the `eig` command in MATLAB) and then discarding the non-imaginary eigenvalues seems obvious. However, due to rounding errors the computed eigenvalues are not exact. Thus one cannot decide whether an eigenvalue with a negligible real part is really on the imaginary axis or only nearby. The first situation indicates a non-dissipative system and would hence require some action to be taken whereas the latter situation does not.

To circumvent this inconvenience we make use of the skew URV factorization [27], [28]: There exist orthogonal matrices $U, V \in \mathbb{R}^{k,k}$ such that the products $T := U^T M V$, and $Z := V^T N V$ are anti triangular (here T is called anti triangular if $t_{ij} = 0$ whenever $i+j \leq n$) whereas the product $S := U^T N U$ is quasi anti triangular, i.e., it is block anti triangular with 1×1 and 2×2 blocks on the anti diagonal. Then the eigenvalues of $\lambda N - M$ can be read off the blocks on the antidiagonals of S, T, Z . More precisely, 1×1 blocks give rise to an eigenvalue pair

$$\lambda_{2i;2i+1} = \pm \sqrt{(t_{i,n+1-i} t_{n+1-i,i}) / (s_{n+1-i,i} z_{n+1-i,i})}. \quad (41)$$

Note that $\lambda_{2i;2i+1}$ form a real or a purely imaginary eigenvalue pair (depending on the signs of $t_{i,n+1-i}$, $t_{n+1-i,i}$, $s_{n+1-i,i}$, and $z_{n+1-i,i}$). The 2×2 blocks are not of interest here, since they correspond to a quadruple of complex non-real, non-imaginary eigenvalues [27], [28]. We will assume that U, V were chosen such that the 1×1 blocks that are responsible for imaginary eigenvalues appear in the upper-right and lower-left corners. (This is always possible. The potentially necessary transformations can be achieved by an eigenvalue reordering algorithm [29].) Then T, S, Z can be partitioned into

$$T = \begin{bmatrix} 0 & 0 & T_{13} \\ 0 & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}, S = \begin{bmatrix} 0 & 0 & -S_{31}^T \\ 0 & S_{22} & -S_{32}^T \\ S_{31} & S_{32} & S_{33} \end{bmatrix},$$

and Z analogously to S where $T_{13}, T_{31}, S_{31}, Z_{31} \in \mathbb{R}^{\vartheta, \vartheta}$ are anti triangular. Partition $U = [U_1, U_2, U_3]$, $V = [V_1, V_2, V_3]$ conformably, that is, $U_1, U_3, V_1, V_3 \in \mathbb{R}^{k, \vartheta}$.

From the definitions of T, S, Z we have $MV = UT$, $MU = VT^T$, $NU = US$, and $NV = VZ$ and looking only at the first block column gives $M[V_1, U_1] = [V_3, U_3] \tilde{M}$, $N[V_1, U_1] = [V_3, U_3] \tilde{N}$ where

$$\tilde{M} = \begin{bmatrix} 0 & T_{13}^T \\ T_{31} & 0 \end{bmatrix}, \tilde{N} = \begin{bmatrix} Z_{31} & 0 \\ 0 & S_{31} \end{bmatrix}.$$

The eigenvalues of the pencil $\lambda \tilde{N} - \tilde{M}$ are given by (41) for $i = 1, \dots, \vartheta$. We note that $\lambda_{2i}, \lambda_{2i+1}$ depend only on the anti diagonal elements of T, S, Z and (depending on their signs) form a real or a purely imaginary eigenvalue pair.

Denote the eigenvectors of $\lambda\tilde{N} - \tilde{M}$ corresponding to the eigenvalues $\lambda_{2i;2i+1}$ by $\tilde{x}_{2i;2i+1}$. Then eigenvectors of $\lambda N - M$ are given by $x_{2i;2i+1} = [V_1, U_1]\tilde{x}_{2i;2i+1}$, because

$$\begin{aligned} Mx_{1;2} &= M[V_1, U_1]\tilde{x}_{1;2} = [V_3, U_3]\tilde{M}\tilde{x}_{1;2} \\ &= \lambda_{1;2}[V_3, U_3]\tilde{N}\tilde{x}_{1;2} = \lambda_{1;2}N[V_1, U_1]\tilde{x}_{1;2} = \lambda_{1;2}Nx_{1;2}. \end{aligned}$$

Note that, although the columns of $[V_1, U_1]$ are not orthogonal in general, there cannot be cancellation in the computation of $x_{2i;2i+1} = [V_1, U_1]\tilde{x}_{2i;2i+1}$, because $\tilde{x}_{2i;2i+1}$ can be chosen such that its first ϑ elements are real whereas the trailing ϑ elements are imaginary.

VII. NUMERICAL EXPERIMENTS

In this section we test Algorithm 8 on four different examples. The first two are the toy example from [1] and [14]. The other two were provided by *CST AG, Darmstadt* and stem from a realization procedure of scattering parameters.

As explained in Subsection VI-B, we can control which of the block entries in the matrix M from (8) can be perturbed through the selection of the basis (25). In our examples we are not going to perturb the matrices Q , S , and R of the supply functional (2) but only (some of) the matrices A , B , C , and D of the system (1). As input parameters of Algorithm 8 we chose $\text{MAXITER} = 1000$ and $\xi = \infty$, unless specified otherwise explicitly.

A. A toy example

In [1] a system of the form (1) with

$$A = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}, \quad B = C^T = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad D = \frac{1}{2}, \quad (42)$$

and $E = I$ was considered together with the supply functional (4). The spectral plot of (42) is given by the solid line in Fig. 5. In [1] the value $\tilde{\omega}$ is chosen as in (38) with $\tau \in (0, \frac{1}{2}]$ and only perturbations to the matrix C are allowed. The left column of Table I shows the results of Algorithm 8 under the same conditions (i.e., employing (38) and restricting the perturbation to C). It is not surprising, that the obtained perturbation norms are very similar to those in [1]. However, allowing perturbations to the matrices B and C results in smaller norms of the perturbation.

TABLE I
INFLUENCE OF THE PARAMETER τ IN (38) ON THE PERFORMANCE OF DISSIPATIVITY ENFORCEMENT ALGORITHM

τ	Iterations	$\frac{\ \Delta C\ }{\ C\ }$	Iterations	$\frac{\sqrt{\ \Delta C\ ^2 + \ \Delta B\ ^2}}{\ C\ }$
0.1	57	0.0661	58	0.0475
0.2	19	0.0661	18	0.0475
0.25	4	0.0661	4	0.0475
0.255	2	0.0661	2	0.0476
0.26	1	0.0670	2	0.0476
0.27	1	0.0696	1	0.0492
0.28	1	0.0722	1	0.0510
0.3	1	0.0773	1	0.0547
0.4	1	0.1031	1	0.0729

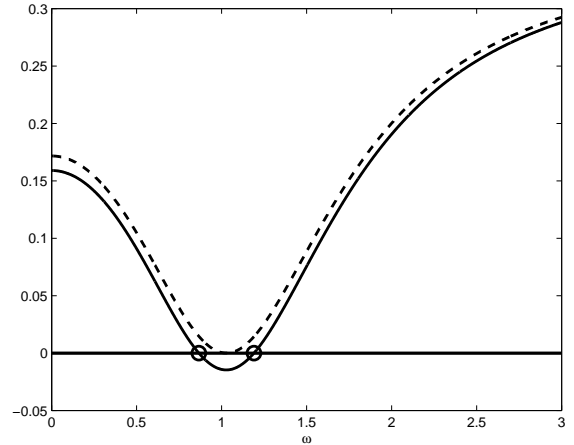


Fig. 5. Spectral plot (solid line) of the system (42) and of the perturbed system (dashed line).

B. A singular toy example

In [14, Example 1] a system of the form (1) with

$$E = \begin{bmatrix} 16 & 12 & -4 & 14 \\ 14 & 8 & 4 & -14 \\ -14 & 8 & -4 & 34 \\ 6 & -4 & 0 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} -0.6 \\ 1 \\ 0.2 \\ -0.3 \end{bmatrix}, \quad (43)$$

$$A = \begin{bmatrix} 6 & -19 & 7 & -9 \\ 11 & 3 & -21 & 18 \\ 25 & -9 & 35 & -16 \\ -27 & 6 & -16 & 38 \end{bmatrix}, \quad C^T = \begin{bmatrix} 3.2 \\ 1.4 \\ 2.6 \\ 1.4 \end{bmatrix},$$

and $D = 0.105$ was considered together with the supply function (3). The spectral plot of the system is given as the solid line in Figure 6. In this example the matrix E is singular and, even more, the pencil $\lambda E - A$ has index 2. However, if one allows for perturbations of the matrix A the index gets reduced to 1 while at the same time a very large eigenvalue at -42849.27 (coming from infinity) is introduced. If one does not allow for perturbations of A the algorithm still works and the passivated system is depicted as the dashed line in Figure 6.

C. A coaxial cable

The example in this section has the form (1) with $E = I$, $n = \rho = 35$, and $m = l = 2$. It stems from a realization procedure of the scattering parameters of a coaxial cable and thus uses the supply functional (4). A spectral plot of the system is given in Fig. 7 at different scales.

A possible perturbed system, which is dissipative, is given in Fig. 8 only for the fine scale, since on the coarse scale it can not be distinguished visually from Fig. 7 (top).

In this example we allowed all of the matrices A , B , C , and D to be perturbed. In Table II we list the ratio of the norms of the computed perturbation and of the original matrices, i.e.

$$\left\| \begin{bmatrix} \Delta A & \Delta B \\ \Delta C & \Delta D \end{bmatrix} \right\| \div \left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\|, \quad (44)$$

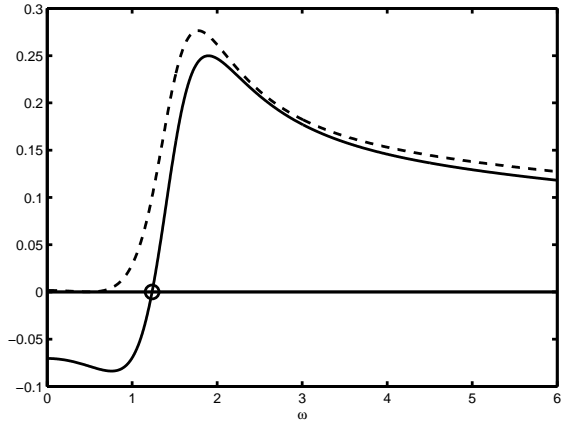


Fig. 6. Spectral plot (solid line) of the system (43) and of the passivated system (dashed line).

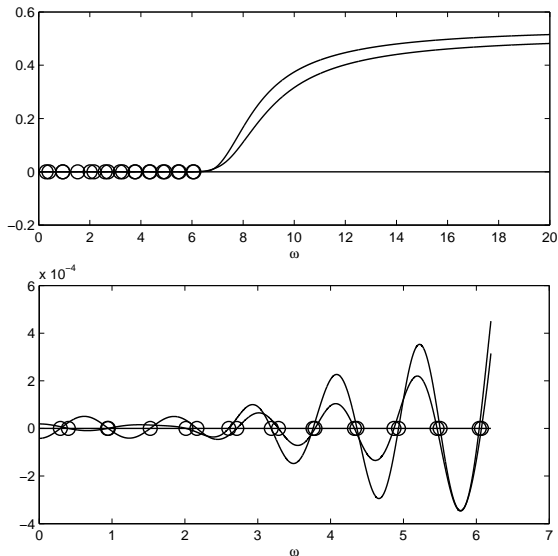


Fig. 7. Spectral plot of the unperturbed coaxial cable at a coarse scales (top) and a fine scale (bottom).

for different choices of $\tilde{\omega}$ and τ as described in Subsection VI-A. All values were computed with $\xi = -0.001$. The obtained results clearly suggest to choose $\tilde{\omega}$ by (37).

D. An RJ45 connector

The example in this section has the form (1) with $E = I$, $n = \rho = 160$, and $m = l = 8$. It stems from a realization procedure of the scattering parameters of an RJ45 connector and thus uses the supply functional (4). Here the matrices A and B have the sparsity structure depicted in Fig. 9, which is typical for matrices coming from a realization procedure, while the matrices C and D are full. To keep the sparsity of Fig. 9 we choose the basis in (25) accordingly.

In Table III we present iteration numbers and perturbation norms for different values of τ . The values in parenthesis include stabilization (with $\xi = -0.001$), the others solely

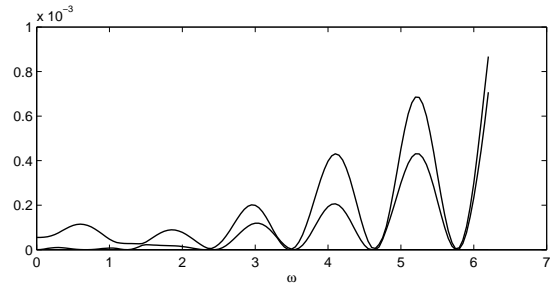


Fig. 8. Spectral plot of the perturbed and dissipative system after 3010 iterations using (37) with $\tau = 10^{-4}$ and perturbations of B and C allowed.

TABLE II
INFLUENCE OF THE CHOICE (38)-(37) AND THE PARAMETER τ ON THE PERFORMANCE OF THE ALGORITHM.

τ	Using (38)		Using (39)		Using (37)	
	Iter.	(44)	Iter.	(44)	Iter.	(44)
0.5	3	1.7953e-4	2	3.8320e-5	1	8.0044e-5
0.2	26	1.0208e-4	17	3.1665e-5	1	3.2018e-5
0.1	55	8.7276e-5	53	2.1890e-5	2	2.6892e-5
0.05	122	5.5879e-5	119	2.0053e-5	6	2.5459e-5
0.02	367	3.2795e-5	316	1.9918e-5	14	2.2585e-5
0.01	646	2.4999e-5	844	2.0255e-5	28	2.0780e-5
0.001	1000	N/A	1000	N/A	298	2.0260e-5

dissipativity enforcement. No parenthesis are given, whenever both variants yielded the same result. "fail" indicates that the maximum iteration count MAXITER=1000 was reached.

TABLE III
INFLUENCE OF PARAMETER τ IN (37) ON PERFORMANCE OF THE ALGORITHM.

τ	Iterations	(44)
5	129 (fail)	2.356e-0 (N/A)
2	87 (fail)	8.384e-1 (N/A)
1	86 (63)	1.539e-1 (1.623e-1)
0.5	38 (32)	8.516e-2 (5.451e-2)
0.2	17	1.755e-2
0.1	19	8.307e-3
0.05	37	4.900e-3
0.02	88	4.737e-3
0.01	112	4.755e-3
0.005	227	4.595e-3
0.002	574	4.562e-3
0.001	874	4.688e-3

Remarkable about the results in Table III is that for large values of $\tau > 0.5$ the number of necessary iterations begins to rise again. This is due to the fact that the example has a more complicated structure, as one can see in Fig. 10, compared with, say, the toy example in Fig. 5. For the RJ45 connector, the large perturbations (which are connected with a large value of τ) can cause lines in the spectral plot, which are already above zero, to move below zero. This then makes a subsequent perturbation necessary, and this process repeats and repeats, similar to Example 9.

VIII. HIGHER-ORDER AND BEHAVIORAL SYSTEMS

For system (1) we introduced the notion of power supply via (2), which is a function of the input value $u(t)$ and the

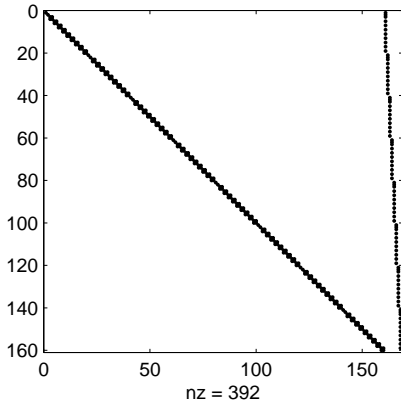


Fig. 9. A Matlab `spy` plot of the concatenated matrix $[A B]$ for the RJ45 connector. The perturbation also only acts on these non-zero entries.

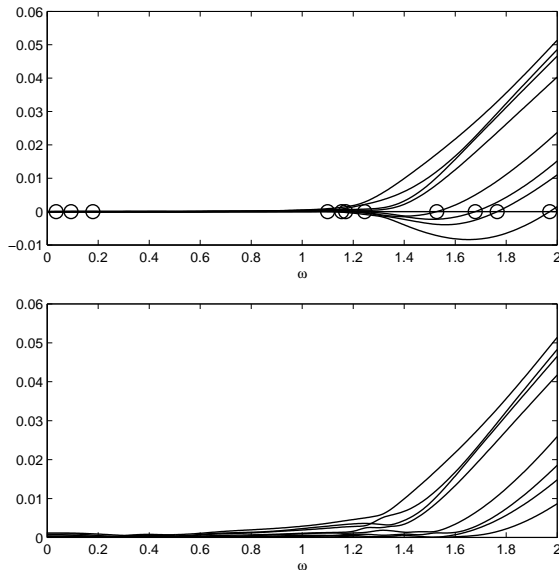


Fig. 10. Top: Spectral plot for the original RJ45 connector. Bottom: The perturbed and dissipative system, using (37) with $\tau = 0.002$.

output value $y(t)$, but not of the state $x(t)$. However, it turns out that this distinction into inputs, outputs, and states is not essential to the problem of dissipativity enforcement. Indeed, one can show that the problem can be formulated (and solved) in a more general framework: the so called *behavior approach*, cf. [20].

Example 10. Consider the electrical circuit given in Fig. 11 with inductance L and resistance R . Using Krichhoff's current law we find that the circuit is described by the equations

$$\begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} I_L \\ I_R \\ I \\ V \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & R & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} I_L \\ I_R \\ I \\ V \end{bmatrix}. \quad (45)$$

There are two possibilities to rewrite this system into a system of the form (1). Assuming that the source in Fig. 11 is voltage-driven results in the state $x = [I_L \ I_R \ I]^T$, the input $u =$

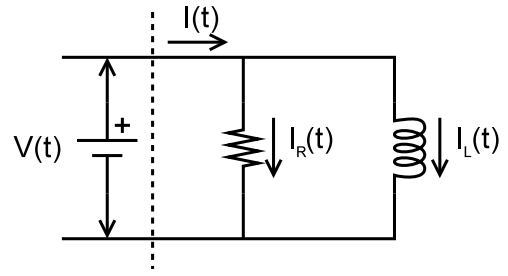


Fig. 11. A simplistic electrical circuit

V , and the output $y = I$. On the other hand, assuming that the source is current-driven yields the state $x = [I_L \ I_R \ V]$, the input $u = I$, and $y = V$. Both choices are legitimate. Sometimes, however, one does not want to make this choice a-priori, in which case it is not reasonable to consider a system of the form (1) with $\rho = n$.

Note that the power supply for the system (45) is measured via $V \cdot I$, which for both choices (voltage-driven or current-driven source) has the form (4).

With the matrices from (1), define

$$F := \begin{bmatrix} 0 & E & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G := \begin{bmatrix} 0 & -A & -B \\ I & -C & -D \end{bmatrix}, \quad z := \begin{bmatrix} y \\ x \\ u \end{bmatrix}, \quad (46)$$

and then $p := \rho + \ell$, and $q := \ell + n + m$ such that $F, G \in \mathbb{R}^{p,q}$ and such that $z \in \mathcal{C}_\infty^q$. Then (u, x, y) is a trajectory of (1) if and only if z fulfills

$$Fz(t) + Gz(t) = 0. \quad (47)$$

Since we can define $P(\lambda) := \lambda F + G \in \mathbb{R}[\lambda]_1^{p,q}$ and rewrite (47) in symbolic fashion as $P(\frac{d}{dt})z = 0$, we call the set of trajectories of (47) the *behavior* of $\lambda F + G$. Furthermore, we call the set of all trajectories of (47) which have compact support the *compact behavior* and denote it by

$$\mathfrak{B}_c(\lambda F + G) := \{z \in \mathcal{C}_\infty^q \mid z \text{ has compact support and fulfills (47)}\}$$

To adopt (2) to the behavioral setting we define

$$H := \begin{bmatrix} Q & 0 & S \\ 0 & 0 & 0 \\ S^T & 0 & R \end{bmatrix}. \quad (48)$$

and then measure the power supply via

$$s(z(t)) = z^T(t)H z(t).$$

Together with the definitions from (46) this gives exactly (2). In the behavior approach, however, it is not necessary to stick to definitions (46) and (48). Arbitrary choices of F , G , and H are possible. An arbitrary pencil $\lambda F + G \in \mathbb{R}[\lambda]_1^{p,q}$ is then called dissipative with respect to an arbitrary symmetric matrix $H = H^T \in \mathbb{R}^{q,q}$ if

$$0 \leq \int_{-\infty}^{\infty} z^*(t)H z(t) dt,$$

for all $z \in \mathfrak{B}_c(\lambda F + G)$. This definition is equivalent to Definition 1 if one uses (46) and (48). In this setting, the equivalent formulation of Theorem 2 states that if the condition

$$\text{rank}(\iota\omega F + G) = p,$$

holds, then the pencil

$$\mathcal{N}(\lambda) := \lambda \begin{bmatrix} 0 & F \\ -F^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & G \\ G^T & H \end{bmatrix}, \quad (49)$$

satisfies

$$\eta(\mathcal{N}(\iota\omega)) = q - p,$$

if and only if $\lambda F + G$ is dissipative with respect to H . Thus, if \mathcal{N} as in (49) is regular, we can compute the purely imaginary eigenvalues and associated eigenvectors of this \mathcal{N} and formulate equation (27) and the method in Section VI in the very same way. However, it is not easily possible to formulate the results from Section V in the behavioral setting, which is why we preferred state-space systems (1) in this paper.

For another choice of F , G , and H consider the higher-order behavior system

$$P_r y^{(r)}(t) + \dots + P_1 y^{(1)}(t) + P_0 y(t) = 0, \quad (50)$$

where $P_j \in \mathbb{R}^{p,q}$, $y \in \mathcal{C}_\infty^q$, and $y^{(j)}$ denotes the j -th derivative of y . Through linearization, i.e., by defining

$$F := \begin{bmatrix} I_q & & & \\ & \ddots & & \\ & & I_q & \\ & & & P_r \end{bmatrix}, \quad z := \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(r-1)} \end{bmatrix} \quad (51)$$

$$G := \begin{bmatrix} 0 & -I_q & & \\ & \ddots & \ddots & \\ & & 0 & -I_q \\ P_0 & \cdots & P_{r-2} & P_{r-1} \end{bmatrix},$$

we can again write system (50) as a system of the form (47). A matching power supply for this kind of system is then allowed to respect not only y but all entries in z (as in (51)), which means that for (50) power supplies which assess $y, \dots, y^{(r-1)}$ (in a quadratic fashion) can be handled by our algorithm. A nice property of our dissipativity enforcement method is that by choosing the basis of the perturbation (25) properly, it is possible to keep the structure of the matrices (51) such that the perturbed system can again be interpreted as a system of the form (50).

IX. CONCLUSION

We have introduced a method to enforce dissipativity (which generalizes the concepts of passivity and contractivity) of LTI control systems. Compared to previous approaches our method works for a larger class of systems (with general supply function, and even behavioral systems). Moreover, the class of allowed perturbations is wider and more flexibly adaptable to the users needs: any part of A , B , C , D , Q , R , S can be perturbed and a weighting of the entries is possible. Also the perturbation can be forced to be sparse, making our

method suitable for large systems, higher order systems, and coordinated systems.

Moreover, we have introduced a new way to choose the target positions of the unwanted purely imaginary eigenvalues, which in our experiments outperformed existing choices. Further advantages are that in the new framework the slopes of the eigenvalue curves are obtained as a by-product and that the described way to compute the needed eigenvalues provides a more robust way to find all of them. Finally, our method is able to enforce stability of the system in addition to dissipativity.

Numerical examples of both academic and industrial origin underpin the desirable properties of the method.

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